Adaptive Control for Nonlinear Uncertain Systems with Actuator Amplitude and Rate Saturation Constraints

Alexander Leonessa
Dep. of Mechanical, Materials and Aerospace Engineering
University of Central Florida
Orlando, Florida 32816
aleo@mail.ucf.com

Yannick Morel
Dep. of Mechanical, Materials and Aerospace Engineering
University of Central Florida
Orlando, Florida 32816
ymorel@gmail.com

ABSTRACT

A direct adaptive nonlinear tracking control framework for multivariable nonlinear uncertain systems with actuator amplitude and rate saturation constraints is developed. To guarantee asymptotic stability of the closed-loop tracking error dynamics in the face of amplitude and rate saturation constraints, the adaptive control signal to a given reference (governor or supervisor) system is modified to effectively robustify the error dynamics to the saturation constraints. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.

1 INTRODUCTION

In light of the increasingly complex and highly uncertain nature of dynamical systems requiring controls, it is not surprising that reliable system models for many high performance engineering applications are unavailable. In the face of such high levels of system uncertainty, robust controllers may unnecessarily sacrifice system performance whereas adaptive controllers are clearly appropriate since they can tolerate far greater system uncertainty levels to improve system performance. However, an implicit assumption inherent in most adaptive control frameworks is that the adaptive control law is implemented without any regard to actuator amplitude and rate saturation constraints. Of course, any electromechanical control actuation device is subject to amplitude and/or rate constraints leading to saturation nonlinearities enforcing limitations on control amplitudes and control rates. As a consequence, actuator nonlinearities arise frequently in practice and can severely degrade closed-loop system performance, and in some cases drive the system to instability. These effects are even more pronounced for adaptive controllers which continue to adapt when the feedback loop has been severed due to the presence of actuator saturation causing unstable controller modes to drift, which in turn leads to severe windup effects.

The research literature on adaptive control with actuator saturation effects is rather limited. Notable exceptions include [1–6]. However, the results reported in [1–6] are confined to linear plants with amplitude saturation. Many practical applications involve nonlinear dynamical systems with simultaneous control amplitude and rate saturation. The presence of control rate saturation may further exacerbate the problem of control amplitude saturation. For example, in advanced tactical fighter aircraft with high maneuverability requirements, pilot induced oscillations [7, 8] can cause actuator amplitude and rate saturation in the control surfaces, leading to catastrophic failures.

In this paper we develop a direct adaptive control framework for adaptive tracking of multivariable nonlinear uncertain systems with amplitude and rate saturation constraints. In particular, we extend the Lyapunov-based direct adaptive control framework developed in [9] to guarantee asymptotic stability of the closed-loop tracking system; that is, asymptotic stability with respect to the closed-loop system states associated with the tracking error dynamics, in the face of actuator amplitude and rate saturation constraints. Specifically, a reference (governor or supervisor) dynamical system is constructed to address tracking and regulation by deriving adaptive update laws that guarantee that the error system dynamics are asymptotically stable, and adaptive controller gains are Lyapunov stable. In the case where the actuator amplitude and rate are limited, the adaptive control signal to the reference system is modified to effectively robustify the error dynamics to the saturation constraints, thus guaranteeing asymptotic stability of the error states.

2 ADAPTIVE TRACKING FOR NONLINEAR UNCERTAIN SYSTEMS

In this section we consider the problem of characterizing adaptive feedback tracking control laws for nonlinear uncertain systems. Specifically, we consider the controlled nonlinear uncertain system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + Bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $f : \mathbb{R}^n \to \mathbb{R}^n$, the matrix $B \in \mathbb{R}^{n\times m}$ is of the form $B = \left[0_{m\times(n-m)} \; B_s^T \right]^T$, with $B_s \in \mathbb{R}^{m\times m}$ full rank and such that there exists $\Lambda \in \mathbb{R}^{m\times m}$ for which $B_s\Lambda$ is positive definite. The control input $u(\cdot)$ in (1) is restricted to the class of *admissible controls* such that (1) has a unique solution forward in time. Here, we assume that a *desired* trajectory (command) $x_{\rm d}(t)$, $t \geq 0$, is given and the aim is to determine the control input $u(t)$, $t \geq 0$, so that $\lim_{t\to\infty} \|x(t) - x_{\rm d}(t)\| = 0$. To achieve this, we construct a reference system $\mathcal{G}_{\rm r}$ given by

$$\dot{x}_{\rm r1}(t) = A_{\rm r}x_{\rm r1}(t) + B_{\rm r}r(t), \quad x_{\rm r1}(0) = x_{{\rm r1}_0}, \quad t \geq 0, \quad (2)$$

where $x_{\rm r1}(t) \in \mathbb{R}^n$, $t \geq 0$, is the reference state vector, $r(t) \in \mathbb{R}^m$, $t \geq 0$, is the reference input, and $A_{\rm r} \in \mathbb{R}^{n\times n}$ and $B_{\rm r} \in \mathbb{R}^{n\times m}$ are such that the pair $(A_{\rm r}, B_{\rm r})$ is stabilizable. Now, we design $u(t)$, $t \geq 0$, and a bounded piecewise-continuous reference function $r(t)$, $t \geq 0$, such that $\lim_{t\to\infty} \|x(t) - x_{\rm r1}(t)\| = 0$ and $\lim_{t\to\infty} \|x_{\rm r1}(t) - x_{\rm d}(t)\| = 0$, respectively, so that $\lim_{t\to\infty} \|x(t) - x_{\rm d}(t)\| = 0$. The following result provides a control architecture that achieves tracking error convergence in the case where the dynamics in (1) are known. The case where $\mathcal{G}$ is unknown is addressed in Theorem 2.2. For the statement of this result, define the tracking error $e(t) \triangleq x(t) - x_{\rm r1}(t), t \geq 0$.

Theorem 2.1. *Consider the nonlinear system $\mathcal{G}$ given by (1) and the reference system $\mathcal{G}_{\rm r}$ given by (2). Assume there exists gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_{\rm r}^* \in \mathbb{R}^{m\times m}$, and $F : \mathbb{R}^n \to \mathbb{R}^s$ such that*

$$0 = f(x) + B\Lambda\Theta^* F(x) - A_{\rm r}x, \qquad x \in \mathbb{R}^n, \quad (3)$$
$$0 = B\Lambda\Theta_{\rm r}^* - B_{\rm r}, \qquad x \in \mathbb{R}^n, \quad (4)$$

hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by

$$K = -R_2^{-1}B_{\rm r}^{\rm T}P, \quad (5)$$

where the $n \times n$ positive definite matrix P satisfies

$$0 = A_{\rm r}^{\rm T}P + PA_{\rm r} - PB_{\rm r}R_2^{-1}B_{\rm r}^{\rm T}P + R_1, \quad (6)$$

and $R_1 \in \mathbb{R}^{n\times n}$ and $R_2 \in \mathbb{R}^{m\times m}$ are arbitrary positive-definite matrices. Then the feedback control law

$$u(t) = \Lambda\left(\Theta_1^*\varphi_1(t) + \Theta_{\rm r}^*(t)r(t)\right), \qquad t \geq 0, \quad (7)$$

where

$$\Theta_1^* \triangleq \left[\, \Theta^* \ \Theta_{\rm r}^* \ \Lambda^{\rm T}B^{\rm T} \,\right] \in \mathbb{R}^{m\times(m+n+s)}, \quad (8)$$
$$\varphi_1(t) \triangleq \left[\, F^{\rm T}(x(t)) \ e^{\rm T}(t)K^{\rm T} \ -\tfrac{1}{2}k_\lambda e^{\rm T}(t)P \,\right]^{\rm T} \in \mathbb{R}^{m+n+s}, \quad t \geq 0, \quad (9)$$

with $k_\lambda > 0$, guarantees that the zero solution $e(t) \equiv 0$, $t \geq 0$, of the error dynamics given by

$$\dot{e}(t) = \left(f(x(t)) + Bu(t)\right) - \left(A_{\rm r}x_{\rm r1}(t) + B_{\rm r}r(t)\right), \quad e(0) = x_0 - x_{{\rm r}_0} \triangleq e_0,\ t \geq 0, \quad (10)$$

is globally asymptotically stable.

Proof. Using the feedback control law given by (7), (10) becomes

$$\dot{e}(t) = f(x(t)) + B\Lambda\Theta_1^*\varphi(t) + B\Lambda\Theta_{\rm r}^*r(t) - A_{\rm r}x_{\rm r1}(t) - B_{\rm r}r(t), \qquad e(0) = e_0, \quad t \geq 0, \quad (11)$$

which, using (8) and (9), we can rewrite as

$$\dot{e}(t) = \left(A_{\rm r} + B\Lambda\Theta_{\rm r}^*K - \frac{1}{2}k_\lambda B\Lambda\Lambda^{\rm T}B^{\rm T}P\right)e(t) + \left(f(x(t)) + B\Lambda\Theta^*F(x(t)) - A_{\rm r}x(t)\right) + \left(B\Lambda\Theta_{\rm r}^* - B_{\rm r}\right)r(t), \quad e(0) = e_0,\ t \geq 0. \quad (12)$$

Now, using (3) and (4), it follows from (12) that

$$\dot{e}(t) = (A_{\rm r} + B_{\rm r}K - \frac{1}{2}k_\lambda B\Lambda\Lambda^{\rm T}B^{\rm T}P)e(t), \quad e(0) = e_0,\ t \geq 0. \quad (13)$$

Now consider the Lyapunov function candidate

$$V(e) = e^{\rm T}Pe, \quad (14)$$

where $P > 0$ satisfies (6). Note that $V(0) = 0$ and, since P is positive definite, $V(e) > 0$ for all $e \neq 0$. Now, letting $e(t)$, $t \geq 0$, denote the solution to (19), using (6), it follows from (13) that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\dot{V}(e(t)) = -e^{\rm T}(t)(R_1 + K^{\rm T}R_2K + K_e)e(t) \leq 0, \quad (15)$$

where $K_e \triangleq k_\lambda PB\Lambda\Lambda^{\rm T}B^{\rm T}P$. Hence, the closed-loop system given by (7) and (10) is Lyapunov stable. Furthermore, since $R_1 + K^{\rm T}R_2K + K_e > 0$, it follows from Theorem 4.4 of [10] that $\lim_{t\to\infty} e(t) = 0$, which concludes this proof.

Theorem 2.1 provides sufficient conditions for characterizing tracking controllers for a given nominal nonlinear dynamical system $\mathcal{G}$. In the next result we show how to construct adaptive gains $\Theta(t) \in \mathbb{R}^{m\times s}$, $t \geq 0$, and $\Theta_r(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, for achieving tracking control in the face of system uncertainty. For this result we do *not* require explicit knowledge of the gain matrices Θ^* and $\Theta_{\rm r}^*$; all that is required is the existence of Θ^* and $\Theta_{\rm r}^*$ such that the compatibility relations (3) and (4) hold.

Theorem 2.2. *Consider the nonlinear system $\mathcal{G}$ given by (1) and the reference system $\mathcal{G}_{\rm r}$ given by (2). Assume there exists gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_{\rm r}^* \in \mathbb{R}^{m\times m}$, and function $F : \mathbb{R}^n \to \mathbb{R}^s$, such that (3) and (4) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (5), where $P = \left[\,P_1\ P_2\,\right] > 0$ satisfies (6), with $P_1 \in \mathbb{R}^{n\times(n-m)}$, $P_2 \in \mathbb{R}^{n\times m}$. In addition, let $\Gamma_1 \in \mathbb{R}^{(m+n+s)\times(m+n+s)}$ and $\Gamma_{\rm r2} \in \mathbb{R}^{m\times m}$ be positive definite, and define $\Theta_1^* \triangleq \left[\,\Theta^*\ \Theta_{\rm r}^*\ \Lambda^{\rm T}B^{\rm T}\,\right] \in \mathbb{R}^{m\times(m+n+s)}$. Then the adaptive feedback control law*

$$u(t) = \Lambda\left(\,\Theta_1(t)\varphi(t) + \Theta_{\rm r2}(t)r(t)\,\right), \qquad t \geq 0, \quad (16)$$

where $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}$, $t \geq 0$, and $\Theta_{\rm r2}(t) \in \mathbb{R}^{m\times m}$, $t \geq 0$, are estimates of Θ_1^* and $\Theta_{\rm r}^*$, respectively, with update laws

$$\dot\Theta_1(t) = -P_2^{\rm T}e(t)\varphi_1(t)\Gamma_1, \quad \Theta_1(0) = \Theta_{10},\, t \geq 0, \quad (17)$$
$$\dot\Theta_{\rm r2}(t) = -P_2^{\rm T}e(t)r^{\rm T}(t)\Gamma_{\rm r2},\ \Theta_{\rm r2}(0) = \Theta_{\rm r20}, \quad (18)$$

guarantees that the closed-loop system given by (10), (17)–(18), with control input (16), is Lyapunov stable, and the error $e(t)$, $t \geq 0$, converges to zero asymptotically.

Proof. With $u(t)$, $t \geq 0$, given by (16) it follows from (3) and (4) that the error dynamics $e(t)$, $t \geq 0$, are given by

$$\begin{aligned}\dot e(t) = {} & (A_{\rm r} + B_{\rm r}K - \frac{1}{2}k_\lambda B\Lambda\Lambda^{\rm T}B^{\rm T}P)e(t) \\ & + B\Lambda\left(\Theta_1(t) - \Theta_1^*\right)\varphi_1(t) + B\Lambda\left(\Theta_{\rm r2}(t) - \Theta_{\rm r2}^*\right)r(t), \\ & e(0) = e_0,\ t \geq 0, \quad (19)\end{aligned}$$

Now consider the Lyapunov function candidate

$$\begin{aligned}V(e,\Theta_1,\Theta_{\rm r2}) = {} & e^{\rm T}Pe + {\rm tr}\left(B_{\rm s}\Lambda(\Theta_1 - \Theta_1^*)\Gamma_1^{-1}(\Theta_1^{\rm T} - \Theta_1^{*{\rm T}})\right) \\ & + {\rm tr}\left(B_{\rm s}\Lambda(\Theta_{\rm r} - \Theta_{\rm r}^*)\Gamma_{\rm r}^{-1}(\Theta_{\rm r}^{\rm T} - \Theta_{\rm r}^{*{\rm T}})\right), \quad (20)\end{aligned}$$

where $P > 0$ satisfies (6), Γ_1 and $\Gamma_{\rm r2}$ are positive definite. Note that $V(0,\Theta_1^*,\Theta_{\rm r2}^*) = 0$ and, since P, Γ_1, $\Gamma_{\rm r2}$ and $B_{\rm s}\Lambda$ are positive definite, $V(e,\Theta_1,\Theta_{\rm r2}) > 0$ for all $(e,\Theta_1,\Theta_{\rm r2}) \neq (0,\Theta_1^*,\Theta_{\rm r2}^*)$. Now, letting $e(t)$, $t \geq 0$, denote the solution to (19), using (6), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned}\dot V(e(t),\Theta_1(t),\Theta_{\rm r2}(t)) = {} & e^{\rm T}(t)P\dot e(t) + \dot e^{\rm T}(t)Pe(t) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_1(t) - \Theta_1^*)\Gamma_1^{-1}\dot\Theta_1^{\rm T}(t)\right) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_{\rm r2}(t) - \Theta_{\rm r2}^*)\Gamma_{\rm r2}^{-1}\dot\Theta_{\rm r2}^{\rm T}(t)\right), \\ & t \geq 0, \quad (21) \\ = {} & 2e^{\rm T}(t)PB\Lambda(\Theta_1(t) - \Theta_1^*)\varphi_1(t) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_1(t) - \Theta_1^*)\Gamma_1^{-1}\dot\Theta_1^{\rm T}(t)\right) \\ & + 2e^{\rm T}(t)PB\Lambda(\Theta_{\rm r2}(t) - \Theta_{\rm r2}^*)r(t) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_{\rm r2}(t) - \Theta_{\rm r2}^*)\Gamma_{\rm r2}^{-1}\dot\Theta_{\rm r2}^{\rm T}(t)\right) \\ & e^{\rm T}(t)P(A_{\rm r} + B_{\rm r}K)e(t) - e^{\rm T}(t)K_e e(t) \\ & + e^{\rm T}(t)(A_{\rm r} + B_{\rm r}K)^{\rm T}Pe(t). \quad (22)\end{aligned}$$

Next, using (17), (18) and the fact that $PB = P_2B_{\rm s}$, we obtain

$$\begin{aligned}\dot V(e(t),\Theta_1(t),\Theta_{\rm r2}(t)) = {} & -e^{\rm T}(t)(R_1 + K^{\rm T}R_2K + K_e)e(t) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_1(t) - \Theta_1^*)(\varphi_1(t)e^{\rm T}(t)P_2 + \Gamma_1^{-1}\dot\Theta_1^{\rm T}(t))\right) \\ & + 2{\rm tr}\left(B_{\rm s}\Lambda(\Theta_{\rm r2}(t) - \Theta_{\rm r2}^*)(r(t)e^{\rm T}(t)P_2 + \Gamma_{\rm r2}^{-1}\dot\Theta_{\rm r2}^{\rm T}(t))\right), \\ = {} & -e^{\rm T}(t)(R_1 + K^{\rm T}R_2K + K_e)e(t), \\ & t \geq 0, \quad (23)\end{aligned}$$

hence, the results obtained in Theorem 2.1 are conserved; that is, the closed-loop system given by (10), (16)–(18) is Lyapunov stable, and, as $R_1 + K^{\rm T}R_2K + K_e > 0$, it follows that $\lim_{t\to\infty} e(t) = 0$.

Remark 2.1. *Note that the conditions in Theorem 2.2 imply that $e(t) \to 0$ as $t \to \infty$ and hence it follows from (17) and (18) that $\dot\Theta(t) \to 0$, $\dot\Theta_{\rm r}(t) \to 0$ as $t \to \infty$.*

It is important to note that the adaptive law (16)–(18) does *not* require explicit knowledge of the gain matrices Θ^* and $\Theta_{\rm r}^*$. Furthermore, no specific knowledge of the structure of the nonlinear term $f(x)$ or matrix B are required to apply Theorem 2.2; all that is required is the existence of $F(x)$ and Λ such that the compatibility relations (3) and (4) hold for a given reference system $\mathcal{G}_{\rm r}$.

3 DYNAMIC ADAPTIVE TRACKING FOR NONLINEAR UNCERTAIN SYSTEMS

In this Section, we build upon the results of the Section 2 and construct an adaptive, dynamic controller for system (1), with stability properties identical to that provided by Theorem 2.2.

The control input is now generated by a dynamic compensator of the form

$$\dot x_{\rm c}(t) = w(t), \qquad x_{\rm c}(0) = x_{\rm c0}, \qquad t \geq 0, \quad (24)$$
$$u(t) = x_{\rm c}(t), \quad (25)$$

where $x_{\rm c}(t) \in \mathbb{R}^m, t \geq 0$, is the compensator state, and $w(t) \in \mathbb{R}^m, t \geq 0$. The expression of $w(t), t \geq 0$, leading to an appropriate control input $u(t), t \geq 0$, can be obtained by building upon the control law presented in the previous Section using various techniques. One such technique is backstepping ([11]). Treating the expression of the control law (7) as a desirable form of $u(t), t \geq 0$, (also referred to as virtual command), expressions for $w(t), t \geq 0$, can be derived guaranteeing convergence of $u(t), t \geq 0$, to this desirable form and accounting for the transient error; ultimately, the properties stated in Theorem 2.1 and Theorem 2.2 are conserved.

To account for the compensator state, we modify the reference system (2) as follows,

$$\dot{x}_{\rm r}(t) = \begin{bmatrix} A_{\rm r} & B_{\rm r} \\ 0_{m\times m} & -\tau_{\rm r}^{-1} \end{bmatrix} x_{\rm r}(t) + \begin{bmatrix} 0_{n\times m} \\ \tau_{\rm r}^{-1} \end{bmatrix} r(t), \quad x_{\rm r}(0) = x_{{\rm r}_0}, \quad t \geq 0, \quad (26)$$

where $x_{\rm r}(t) = \left[x_{r1}^{\rm T}(t) \; x_{r2}^{\rm T}(t) \right]^{\rm T}, t \geq 0$, with $x_{r1}(t) \in \mathbb{R}^n$, $x_{r2}(t) \in \mathbb{R}^m, t \geq 0$, and $\tau_{\rm r} \in \mathbb{R}^{m\times m}$ is positive definite.

As mentioned above, the expression of $u(t), t \geq 0$, provided by (7) becomes a desirable form

$$u_{\rm d}^*(t) \triangleq \Lambda\left(\Theta_1^* \varphi_1(t) + \Theta_{\rm r2}^* x_{r2}(t)\right), \qquad t \geq 0, \quad (27)$$

with $r(t), t \geq 0$, in (7) being replaced by $x_{r2}(t), t \geq 0$, to account for the modification to the reference system. With this definition of $u_{\rm d}^*(t), t \geq 0$, the error dynamics (10) becomes

$$\dot{e}(t) = (A_{\rm r} + B_{\rm r}K + K_e)e(t) + B\left(u(t) - u_{\rm d}^*(t)\right), \quad e(0) = e_0, \quad t \geq 0, \quad (28)$$

where $u_{\rm d}^*(t)$, $t \geq 0$, is such that for $u(t) = u_{\rm d}^*(t)$, $t \geq 0$, we can guarantee that $e(t), t \geq 0$, converges to zero, as stated in Theorem 2.1. Defining the error $e_u^*(t) \triangleq u(t) - u_{\rm d}^*(t), t \geq 0$, the remaining problem is to find the appropriate expression for $w(t), t \geq 0$, which we denote $w^*(t), t \geq 0$, such that $e_u^*(t), t \geq 0$, converges to zero.

Note that a number of constant parameters in (27) are uncertain and will be estimated, with appropriate update laws similar to those in Theorem 2.2. Ultimately, the expression we desire $u(t), t \geq 0$, to track is

$$u_{\rm d}(t) = \Lambda\left(\Theta_1(t)\varphi_1(t) + \Theta_{\rm r2}(t)x_{r2}(t)\right), \qquad t \geq 0, \quad (29)$$

where $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}, \Theta_{\rm r2}(t) \in \mathbb{R}^{m\times m}, t \geq 0$, are estimates of Θ_1^* and $\Theta_{\rm r}^*$, respectively.

Backstepping techniques are classically plagued with a well documented issue referred to as "explosion of terms" ([12]). As the derivation of the control law progresses through the backstepping procedure, the expressions involved in the derivations become increasingly expansive, to an extent that the final expression of the control law can become difficult to manage. More specifically, the expression of $w^*(t)$, $t \geq 0$, will in our case include that of $\dot{u}_{\rm d}(t), t \geq 0$, that is, with update laws similar to that from Theorem 2.2, and $\Theta_1(t) = \left[\Theta_{11}(t) \; \Theta_{12}(t) \right], t \geq 0$, with $\Theta_{11}(t) \in \mathbb{R}^{m\times s}, \Theta_{12}(t) \in \mathbb{R}^{m\times m+n}, t \geq 0$,

$$\begin{aligned}\dot{u}_{\rm d}(t) = \Lambda\Big(& -P_2^{\rm T} e(t)\left(\varphi_1^{\rm T}(t)\Gamma_1\varphi_1(t) + x_{\rm r2}^{\rm T}(t)\Gamma_{\rm r2}x_{\rm r2}(t)\right) \\ & + \Theta_{11}(t)\frac{dF(x(t))}{dx(t)}\left(f(x(t)) + Bu(t)\right) \\ & + \Theta_{12}(t)\begin{bmatrix} K \\ -\frac{1}{2}k_\lambda P \end{bmatrix}\dot{e}(t) + \Theta_{\rm r2}(t)\tau_{\rm r}^{-1}\left(x_{\rm r2}(t) - r(t)\right)\Big), \\ & t \geq 0, \quad (30)\end{aligned}$$

with $\Gamma_1 \in \mathbb{R}^{(m+n+s)\times(m+n+s)}$, $\Gamma_{\rm r2} \in \mathbb{R}^{m\times m}$, $\Gamma_1 > 0$, $\Gamma_2 > 0$. Note that the above expression can be rewritten as

$$\dot{u}_{\rm d}(t) = g(t) + h(t)\Theta_2^*\varphi_2(t), \qquad t \geq 0, \quad (31)$$

where

$$h(t) \triangleq \Lambda\Theta_1(t)\begin{bmatrix} \frac{dF(x(t))}{dx(t)} \\ K \\ -\frac{1}{2}k_\lambda P \end{bmatrix}, \quad \Theta_2^* \triangleq B\left[-\Lambda\Theta^* \; I_m \right], (32)$$

$$\varphi_2(t) \triangleq \left[F(x(t))^{\rm T} \; u(t)^{\rm T} \right]^{\rm T}, \qquad t \geq 0, \; (33)$$

and

$$\begin{aligned} g(t) \triangleq \Lambda\Big(& -P_2^{\rm T} e(t)\left(\varphi_1^{\rm T}(t)\Gamma_1\varphi_1(t) + x_{\rm r2}^{\rm T}(t)\Gamma_{\rm r2}x_{\rm r2}(t)\right) \\ & -\Theta_{12}(t)\left[K^{\rm T} \; -\tfrac{1}{2}k_\lambda P \right]^{\rm T}\dot{x}_{\rm r1}(t) \\ & +\Theta_{\rm r2}(t)\tau_{\rm r}^{-1}\left(x_{\rm r2}(t) - r(t)\right)\Big) + h(t)A_{\rm r}x(t), \\ & t \geq 0, \quad (34)\end{aligned}$$

which allows to isolate the unknown term Θ_2^* in $\dot{u}_{\rm d}(t), t \geq 0$.

Next, we build upon the results in Theorem 2.2, and present a control algorithm providing the same stability properties, but for a control input generated by (24)–(25).

Theorem 3.1. *Consider the controlled nonlinear system $\mathcal{G}$ given by (1) and reference system (26). Assume there exist gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_{\rm r}^* \in \mathbb{R}^{m\times m}$, and a function $F : \mathbb{R}^n \to \mathbb{R}^s$, such that (3) and (4) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (5), where $P = \left[P_1 \; P_2 \right] > 0$, with $P_1 \in \mathbb{R}^{n\times(n-m)}$, $P_2 \in \mathbb{R}^{n\times m}$, satisfies (6), and define $\Theta_1^* \triangleq \left[\Theta^* \; \Theta_{\rm r}^* \; \Lambda^{\rm T}B^{\rm T} \right] \in \mathbb{R}^{m\times(m+n+s)}$, $\Theta_2^* \triangleq B\left[-\Lambda\Theta^* \; I_m \right] \in \mathbb{R}^{n\times(m+s)}$, and $\Theta_3^* \triangleq B^{\rm T} \in \mathbb{R}^{m\times n}$. Consider the control input $u(t), t \geq 0$, generated by (24)–(25), where*

$$w(t) = g(t) + h(t)\Theta_2(t)\varphi_2(t) - 2\Theta_3(t)Pe(t) - K_u e_u(t), \qquad t \geq 0, \quad (35)$$

with $h(t) \in \mathbb{R}^{m\times n}, t \geq 0$, given by (33), $g(t) \in \mathbb{R}^m, t \geq 0$, given by (34), $\varphi_2(t) \triangleq \left[F(x(t))^{\mathrm{T}}\ u(t)^{\mathrm{T}}\right]^{\mathrm{T}}$, $K_u \in \mathbb{R}^{m\times m}$ is positive definite, and $\Theta_2(t)$, $\Theta_3(t), t \geq 0$, are estimates of Θ_2^* and Θ_3^*, respectively. The tracking errors are defined as

$$e(t) \triangleq x(t) - x_{\mathrm{r}1}(t), \qquad e_u(t) \triangleq u(t) - u_{\mathrm{d}}(t), \quad t \geq 0, \tag{36}$$

where

$$u_{\mathrm{d}}(t) = \Lambda\left(\Theta_1(t)\varphi_1(t) + \Theta_{\mathrm{r}2}(t)x_{r2}(t)\right), \qquad t \geq 0, \tag{37}$$
$$\varphi_1(t) \triangleq \left[F^{\mathrm{T}}(x(t))\ e^{\mathrm{T}}(t)K^{\mathrm{T}}\ -\tfrac{1}{2}k_\lambda e^{\mathrm{T}}(t)P\right]^{\mathrm{T}}, \tag{38}$$

with $\varphi_1(t) \in \mathbb{R}^{m+n+s}, t \geq 0$, and $\Theta_1(t)$, $\Theta_{\mathrm{r}2}(t), t \geq 0$, are estimates of Θ_1^* and Θ_{r}^*, respectively. These estimates $\Theta_1(t) \in \mathbb{R}^{m\times(m+n+s)}, \Theta_{\mathrm{r}2}(t) \in \mathbb{R}^{m\times m}$, $\Theta_2(t) \in \mathbb{R}^{n\times(m+s)}$ and $\Theta_3(t) \in \mathbb{R}^{m\times n}, t \geq 0$, are obtained as follows

$$\dot{\Theta}_1(t) = -P_2^{\mathrm{T}}e(t)\varphi_1^{\mathrm{T}}(t)\Gamma_1, \qquad \Theta_1(0) = \Theta_{10},\ t \geq 0, \tag{39}$$
$$\dot{\Theta}_{\mathrm{r}2}(t) = -P_2^{\mathrm{T}}e(t)x_{\mathrm{r}2}^{\mathrm{T}}(t)\Gamma_{\mathrm{r}2}, \qquad \Theta_{\mathrm{r}2}(0) = \Theta_{\mathrm{r}20}, \tag{40}$$
$$\dot{\Theta}_2(t) = -h(t)^{\mathrm{T}}e_u(t)\varphi_2(t)^{\mathrm{T}}\Gamma_2,\ \Theta_2(0) = \Theta_{20}, \tag{41}$$
$$\dot{\Theta}_3(t) = e_u(t)e^{\mathrm{T}}(t)P\Gamma_3, \qquad \Theta_3(0) = \Theta_{30}, \tag{42}$$

where $\Gamma_1 \in \mathbb{R}^{(m+n+s)\times(m+n+s)}$, $\Gamma_{\mathrm{r}2} \in \mathbb{R}^{m\times m}$, $\Gamma_2 \in \mathbb{R}^{(m+s)\times(m+s)}$, and $\Gamma_3 \in \mathbb{R}^{n\times n}$, are positive definite.

Then, the control input $u(t), t \geq 0$, generated by (35), guarantees that the closed-loop system given by (10), (39)–(42), with control input generated by (24)–(25) with (35), is Lyapunov stable, and the errors $e(t), e_u(t), t \geq 0$, converge to the origin, asymptotically.

Proof. From (36) and (37), we have

$$u(t) = \Lambda\left(\Theta_1(t)\varphi_1(t) + \Theta_{\mathrm{r}2}(t)x_{r2}(t)\right) + e_u(t), \quad t \geq 0, \tag{43}$$

which we expand, using (38), into

$$\begin{aligned} u(t) = {} & \Lambda\left(\Theta^* F(x(t)) + \Theta_{\mathrm{r}}^*(x_{\mathrm{r}2}(t) + Ke(t))\right) \\ & -\frac{1}{2}k_\lambda \Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}Pe(t) + \Lambda(\Theta_1(t) - \Theta_1^*)\varphi_1(t) \\ & +\Lambda(\Theta_{\mathrm{r}2}(t) - \Theta_{\mathrm{r}2}^*)x_{\mathrm{r}2}(t) + e_u(t), \qquad t \geq 0. \end{aligned} \tag{44}$$

Substituting (3), (4), and (44) in (10), we obtain

$$\begin{aligned} \dot{e}(t) = {} & (A_{\mathrm{r}} + B_{\mathrm{r}}K - \frac{1}{2}k_\lambda\Lambda\Lambda^{\mathrm{T}}B^{\mathrm{T}}P)e(t) \\ & +B\Lambda(\Theta_1(t) - \Theta_1^*)\varphi_1(t) + B\Lambda(\Theta_{\mathrm{r}2}(t) - \Theta_{\mathrm{r}2}^*)x_{\mathrm{r}2}(t) \\ & +Be_u(t), \qquad e(0) = e_0,\ t \geq 0, \end{aligned} \tag{45}$$

Similarly, from (24), (25), (31) and (36),

$$\dot{e}_u(t) = w(t) - g(t) - h(t)\Theta_2^*\varphi_2(t),\ e_u(0) = e_{u0},\ t \geq 0, \tag{46}$$

which, using (35) and $\Theta_3^* = B^{\mathrm{T}}$, can be rewritten as

$$\begin{aligned} \dot{e}_u(t) = {} & -2B^{\mathrm{T}}Pe(t) - K_u e_u(t) + h(t)(\Theta_2(t) - \Theta_2^*)\varphi_2(t) \\ & +2(\Theta_3^* - \Theta_3(t))Pe(t), \qquad e_u(0) = e_{u0},\ t \geq 0. \end{aligned} \tag{47}$$

Now consider the Lyapunov function candidate

$$\begin{aligned} & V(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_{\mathrm{r}2}) = \\ & \quad e^{\mathrm{T}}Pe + \frac{1}{2}e_u^{\mathrm{T}}e_u + \mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_1 - \Theta_1^*)\Gamma_1^{-1}(\Theta_1^{\mathrm{T}} - \Theta_1^{*\mathrm{T}})\right) \\ & \quad +\mathrm{tr}\left((\Theta_2 - \Theta_2^*)\Gamma_2^{-1}(\Theta_2^{\mathrm{T}} - \Theta_2^{*\mathrm{T}})\right) \\ & \quad +\mathrm{tr}\left((\Theta_3 - \Theta_3^*)\Gamma_3^{-1}(\Theta_3^{\mathrm{T}} - \Theta_3^{*\mathrm{T}})\right) \\ & \quad +\mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_{\mathrm{r}2} - \Theta_{\mathrm{r}2}^*)\Gamma_{\mathrm{r}2}^{-1}(\Theta_{\mathrm{r}2}^{\mathrm{T}} - \Theta_{\mathrm{r}2}^{*\mathrm{T}})\right), \end{aligned} \tag{48}$$

where $P > 0$ satisfies (6). Note that $V(0,0,\Theta_1^*,\Theta_2^*,\Theta_3^*,\Theta_{\mathrm{r}2}^*) = 0$ and, since P, Γ_1, Γ_2, Γ_3, $\Gamma_{\mathrm{r}2}$ and $B_{\mathrm{s}}\Lambda$ are positive definite, $V(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_{\mathrm{r}2}) > 0$ for all $(e, e_u, \Theta_1, \Theta_2, \Theta_3, \Theta_{\mathrm{r}2}) \neq (0, 0, \Theta_1^*, \Theta_2^*, \Theta_3^*, \Theta_{\mathrm{r}2}^*)$. Now, using (6), (39)–(42), and (36), it follows that the Lyapunov derivative along the closed-loop system trajectories is given by

$$\begin{aligned} & \dot{V}(e(t), e_u(t), \Theta_1(t), \Theta_2(t), \Theta_3(t), \Theta_{\mathrm{r}2}(t)) = \\ & \qquad e^{\mathrm{T}}(t)P\dot{e}(t) + \dot{e}^{\mathrm{T}}(t)Pe(t) + e_u(t)^{\mathrm{T}}\dot{e}_u(t) \\ & \qquad +2\mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_1(t) - \Theta_1^*)\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t)\right) \\ & \qquad +2\mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_{\mathrm{r}2}(t) - \Theta_{\mathrm{r}2}^*)\Gamma_{\mathrm{r}2}^{-1}\dot{\Theta}_{\mathrm{r}2}^{\mathrm{T}}(t)\right) \\ & \qquad +2\mathrm{tr}\left((\Theta_2(t) - \Theta_2^*)\Gamma_2^{-1}\dot{\Theta}_2^{\mathrm{T}}(t)\right) \\ & \qquad +2\mathrm{tr}\left((\Theta_3(t) - \Theta_3^*)\Gamma_3^{-1}\dot{\Theta}_3^{\mathrm{T}}(t)\right),\ t \geq 0, \\ & = e^{\mathrm{T}}(t)P(A_{\mathrm{r}} + B_{\mathrm{r}}K)e(t) + e^{\mathrm{T}}(t)(A_{\mathrm{r}} + B_{\mathrm{r}}K)^{\mathrm{T}}Pe(t) \\ & \quad -e^{\mathrm{T}}(t)K_e e(t) - e_u^{\mathrm{T}}(t)K_u e_u(t) \\ & \quad +2\mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_1(t) - \Theta_1^*)\left(\Gamma_1^{-1}\dot{\Theta}_1^{\mathrm{T}}(t) + \varphi_1(t)e^{\mathrm{T}}(t)P_2\right)\right) \\ & \quad +2\mathrm{tr}\left(B_{\mathrm{s}}\Lambda(\Theta_{\mathrm{r}2}(t) - \Theta_{\mathrm{r}2}^*)\left(\Gamma_{\mathrm{r}2}^{-1}\dot{\Theta}_{\mathrm{r}2}^{\mathrm{T}}(t) + x_{\mathrm{r}2}(t)e^{\mathrm{T}}(t)P_2\right)\right) \\ & \quad +2\mathrm{tr}\left((\Theta_2(t) - \Theta_2^*)\left(\Gamma_2^{-1}\dot{\Theta}_2^{\mathrm{T}}(t) + \varphi_2(t)e_u^{\mathrm{T}}(t)h(t)\right)\right) \\ & \quad +2\mathrm{tr}\left((\Theta_3(t) - \Theta_3^*)\left(\Gamma_3^{-1}\dot{\Theta}_3^{\mathrm{T}}(t) - Pe(t)e_u^{\mathrm{T}}(t)\right)\right), \\ & = -e^{\mathrm{T}}(t)(R_1 + K^{\mathrm{T}}R_2K + K_e)e(t) - e_u^{\mathrm{T}}(t)K_u e_u(t), \end{aligned} \tag{49}$$

hence, the closed-loop system given by (10), (46), (39)–(42) is Lyapunov stable. Furthermore, since $R_1 + K^{\mathrm{T}}R_2K + K_e > 0$ and $K_u > 0$, it follows from Theorem 4.4 of [10] that $\lim_{t\to\infty} e(t) = 0$, and $\lim_{t\to\infty} e_u(t) = 0$.

Remark 3.1. *Note that a parallel can be drawn between*

(24) and the actuator dynamics of a physical system. The form of (24) was chosen to be an integrator for simplicity, but it can be readily modified to represent the actuator dynamics of a considered system. Hence, the presented approach can allow to elegantly account for actuator dynamics in the control framework.

4 ADAPTIVE TRACKING WITH ACTUATOR AMPLITUDE AND RATE SATURATION CONSTRAINTS

In this section we extend the adaptive control framework presented in Section 3 to account for actuator amplitude and rate saturation constraints. Recall that Theorem 2.2 guarantees convergence of the tracking error $e(t)$, $t \geq 0$, to a neighborhood of zero; that is, the state vector $x(t)$, $t \geq 0$, converges to a neighborhood of the reference state vector $x_{\mathrm{r1}}(t)$, $t \geq 0$. Furthermore, it is important to note that the compensator state $w(t)$, $t \geq 0$, given by (35), depends on the *reference input* $r(t)$, $t \geq 0$, through $\dot{x}_{\mathrm{r2}}(t), t \geq 0$. Since for a fixed set of initial conditions there exists a one-to-one mapping between the reference input $r(t)$, $t \geq 0$, and the reference state $x_{\mathrm{r1}}(t)$, $t \geq 0$, it follows that the control signal in (16) guarantees convergence of the state $x(t)$, $t \geq 0$, to a neighborhood of the reference state $x_{\mathrm{r1}}(t)$, $t \geq 0$, corresponding to the specified reference input $r(t)$, $t \geq 0$. Of course, the reference input $r(t)$, $t \geq 0$, should be chosen so as to guarantee asymptotic convergence to a *desired* state vector $x_{\mathrm{d}}(t)$, $t \geq 0$. However, the choice of such a reference input $r(t)$, $t \geq 0$, is not unique since the reference state vector $x_{\mathrm{r1}}(t)$, $t \geq 0$, can converge to the desired state vector $x_{\mathrm{d}}(t)$, $t \geq 0$, without matching its transient behavior.

Next, we provide a framework wherein we construct a family of reference inputs $r(t)$, $t \geq 0$, with associated reference state vectors $x_{\mathrm{r1}}(t)$, $t \geq 0$, that guarantee that a given reference state vector within this family converges to a desired state vector $x_{\mathrm{d}}(t)$, $t \geq 0$, in the face of actuator amplitude and rate saturation constraints.

From (24) and (25), it is clear that $\dot{u}(t), t \geq 0$, is explicitly depending on $w(t), t \geq 0$, which itself depends upon the reference signal $r(t), t \geq 0$. More specifically, from (24), (25), (34) and (35),

$$\begin{aligned} \dot{u}(t) &= H(s(t), r), \qquad t \geq 0, \\ &= g_1(t) + h(t)\Theta_2(t)\varphi_2(t) - 2\Theta_3(t)Pe(t) - K_u e_u(t) \\ &\quad -\Lambda\Theta_{\mathrm{r2}}(t)\tau_{\mathrm{r}}^{-1} r(t), \end{aligned} \tag{50}$$

where $s(t) \triangleq (x(t), x_{\mathrm{r}}(t), \Theta_{\mathrm{r2}}(t), \Theta_2(t), \Theta_3(t), e(t), e_u(t))$, $t \geq 0$, and

$$\begin{aligned} g_1(t) \triangleq \Lambda\Big(&-P_2^{\mathrm{T}} e(t)\left(\varphi_1^{\mathrm{T}}(t)\Gamma_1\varphi_1(t) + x_{\mathrm{r2}}^{\mathrm{T}}(t)\Gamma_{\mathrm{r2}} x_{\mathrm{r2}}(t)\right) \\ &-\Theta_1(t)\left[\, 0_{n\times s}\; K^{\mathrm{T}} \; -\tfrac{1}{2}k_\lambda P \,\right]^{\mathrm{T}} \dot{x}_{\mathrm{r1}}(t) \\ &+\Theta_{\mathrm{r2}}(t)\tau_{\mathrm{r}}^{-1} x_{\mathrm{r2}}(t)\Big) + h(t)A_{\mathrm{r}} x(t), \; t \geq 0. \end{aligned} \tag{51}$$

Using (50), the reference input $r(t)$, $t \geq 0$, can be expressed as

$$\begin{aligned} r(t) &= H^{-1}(s(t), \dot{u}(t)), \qquad t \geq 0, \\ &= \tau_{\mathrm{r}}\Theta_{\mathrm{r2}}^{-1}\Lambda^{-1}(g_1(t) + h(t)\Theta_2(t)\varphi_2(t) - 2\Theta_3(t)Pe(t) \\ &\quad -K_u e_u(t) - \dot{u}(t)). \end{aligned} \tag{52}$$

The above expression relates the reference input to the time rate of change of the control input.

Next, we assume that the control signal is amplitude and rate limited so that $|u_i(t)| \leq u_{\max}$ and $|\dot{u}_i(t)| \leq \dot{u}_{\max}$, $t \geq 0$, $i = 1, \ldots, m$, where $u_i(t)$ and $\dot{u}_i(t)$ denote the ith component of $u(t)$ and $\dot{u}(t)$, respectively, and $u_{\max} > 0$ and $\dot{u}_{\max} > 0$ are given. For the statement of our main result the following definitions are needed. For $i \in \{1, \cdots, m\}$ define

$$\sigma(u_i(t), \dot{u}_i(t)) \triangleq \begin{cases} 0 & \text{if } |u_i(t)| = u_{\max} \text{ and } u_i(t)\dot{u}_i(t) > 0, \\ 1 & \text{otherwise,} \end{cases} \quad t \geq 0, \tag{53}$$

$$\sigma^*(u_i(t), \dot{u}_i(t)) \triangleq \min\left\{\sigma(u_i(t), \dot{u}_i(t)), \frac{\dot{u}_{\max}}{|\dot{u}_i(t)|}\right\}, \; t \geq 0. \tag{54}$$

Note that for $i \in \{1, \cdots, m\}$ and $t = t_1 > 0$, the function $\sigma^*(\cdot,\cdot)$ is such that the following properties hold:

i) If $|u_i(t_1)| = u_{\max}$ and $u_i(t_1)\dot{u}_i(t_1) > 0$, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1), \dot{u}_i(t_1)) = 0$.
ii) If $|\dot{u}_i(t_1)| > \dot{u}_{\max}$ and $|u_i(t_1)| < u_{\max}$ or if $|\dot{u}_i(t_1)| > \dot{u}_{\max}$ and $|u_i(t_1)| = u_{\max}$ and $u_i(t_1)\dot{u}_i(t_1) \leq 0$, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1), \dot{u}_i(t_1)) = \dot{u}_{\max}\mathrm{sgn}(\dot{u}_i(t_1))$, where $\mathrm{sgn}\,\dot{u}_i \triangleq |\dot{u}_i|/\dot{u}_i$.
iii) If no constraint is violated, then $\dot{u}_i(t_1)\sigma^*(u_i(t_1), \dot{u}_i(t_1)) = \dot{u}_i(t_1)$.

Finally, we define the component decoupled diagonal nonlinearity $\Sigma(u, \dot{u})$ by

$$\begin{aligned} \Sigma(u(t), \dot{u}(t)) \triangleq \mathrm{diag}[&\sigma^*(u_1(t), \dot{u}_1(t)), \sigma^*(u_2(t), \dot{u}_2(t)), \ldots \\ &\ldots, \sigma^*(u_m(t), \dot{u}_m(t))], \quad t \geq 0. \end{aligned} \tag{55}$$

Theorem 4.1. *Consider the controlled nonlinear system $\mathcal{G}$ given by (1) and reference system (26). Assume there exist gain matrices $\Theta^* \in \mathbb{R}^{m\times s}$ and $\Theta_{\mathrm{r}}^* \in \mathbb{R}^{m\times m}$, and a function $F : \mathbb{R}^n \to \mathbb{R}^s$, such that (3) and (4) hold. Furthermore, let $K \in \mathbb{R}^{m\times n}$ be given by (5), where $P > 0$ satisfies (6). In addition, for a given desired reference input $r_{\mathrm{d}}(t)$, $t \geq 0$, let* *the reference input $r(t)$, $t \geq 0$, be given by*

$$r(t) = H^{-1}(s(t), \Sigma(u(t), \dot{u}^*(t))\dot{u}^*(t)), \qquad t \geq 0, \tag{56}$$

where $s(t) = (x(t), x_{\rm r}(t), \Theta_{\rm r2}(t), \Theta_2(t), e(t), e_u(t))$, $t \geq 0$, and $\dot{u}^*(t) \triangleq H(s(t), r_{\rm d}(t)), t \geq 0$. Then the adaptive feedback control law (35), with update laws (39)–(42) and reference input $r(t)$, $t \geq 0$, provided by (56) guarantees

- *i) asymptotic convergence of $(e(t), e_u(t)), t \geq 0$, to the origin.*
- *ii) $|u_i(t)| \leq u_{\max}$ for all $t \geq 0$ and $i = 1, \ldots, m$.*
- *iii) $|\dot{u}_i(t)| \leq \dot{u}_{\max}$ for all $t \geq 0$ and $i = 1, \ldots, m$.*

Proof. *i*) is a direct consequence of Theorem 3.1 with $r(t)$, $t \geq 0$, given by (56). To prove *ii*) and *iii*) note that it follows from (50), (52), and (56) that

$$\begin{aligned}\dot{u}(t) &= H(s(t), \dot{r}(t)) = H(s(t), H^{-1}(s(t), \Sigma(u(t), \dot{u}^*(t))\dot{u}^*(t))) \\ &= \Sigma(u(t), \dot{u}^*(t))\dot{u}^*(t), \qquad t \geq 0,\end{aligned} \tag{57}$$

which implies $\dot{u}_i(t) = \sigma^*(u_i(t), \dot{u}_i^*(t))\dot{u}_i^*(t)$, $i = 1, \cdots, m$, $t \geq 0$. Hence, if the control input $u_i(t), t \geq 0$, with a rate of change $\dot{u}_i^*(t)$, $i = 1, \cdots, m$, $t \geq 0$, does not violate the amplitude and rate saturation constraints, then it follows from (54) that $\sigma^*(u_i(t), \dot{u}_i^*(t)) = 1$ and $\dot{u}_i(t) = \dot{u}_i^*(t)$, $i = 1, \cdots, m$, $t \geq 0$. Alternatively, if the pair $(u_i(t), \dot{u}_i^*(t))$, $i = 1, \cdots, m$, $t \geq 0$, violates one or more of the input amplitude and/or rate constraints, then (53), (54), and (57) imply

- *i)* $\dot{u}_i(t) = 0$ for all $t \geq 0$ if $|u_i(t)| = u_{\max}$ and $u_i(t)\dot{u}_i^*(t) > 0$; and
- *ii)* $\dot{u}_i(t) = \dot{u}_{\max}{\rm sgn}(\dot{u}_i^*(t))$ for all $t \geq 0$ if $|\dot{u}_i^*(t)| > \dot{u}_{\max}$ and $|u_i(t)| < u_{\max}$ or if $|\dot{u}_i^*(t)| > \dot{u}_{\max}$ and $|u_i(t)| = u_{\max}$ and $u_i(t)\dot{u}_i(t) \leq 0$;

which, for $u_i(0) \leq u_{\max}$, guarantee that $|u_i(t)| \leq u_{\max}$ and $|\dot{u}_i(t)| \leq \dot{u}_{\max}$ for all $t \geq 0$ and $i = 1, \cdots, m$.

Remark 4.1. *In accordance with (24)–(25), the value of $\dot{u}(t), t \geq 0$, used extensively in the above proceedings, can be obtained from the value of the time derivative of the compensator state $x_{\rm c}(t), t \geq 0$, or that of $w(t), t \geq 0$, since, from (24)–(25), $\dot{u}(t) = \dot{x}_{\rm c}(t) = w(t), t \geq 0$.*

Note that it follows from Theorem 4.1 that if the desired reference input $r_{\rm d}(t)$, $t \geq 0$, is such that the actuator amplitude and/or rate saturation constraints are not violated, then $r(t) = r_{\rm d}(t)$, $t \geq 0$, and hence $x(t)$, $t \geq 0$, converges to a neighborhood of $x_{\rm d}(t)$, $t \geq 0$. Alternatively, if there exists $t = t^* > 0$ such that the desired reference input drives one or more of the control inputs to the saturation boundary, then $r(t) \neq r_{\rm d}(t)$, $t > t^*$. However, as long as the time interval over which the control input remains saturated is finite, the reference signal ultimately reverts to its desired value, and tracking properties are preserved. Of course, if there exists a solution to the tracking problem wherein the input amplitude and rate saturation constraints are not violated when the tracking error is within certain bounds, then our approach is guaranteed to work.

5 ILLUSTRATIVE NUMERICAL EXAMPLE

In this section we present a numerical example to demonstrate the utility of the proposed direct adaptive control framework for adaptive stabilization in the face of actuator amplitude and rate saturation constraints. Note that a tracking example is not provided due to lack of space, but the presented approach is equally relevant to such problems.

Example 5.1. *Consider the nonlinear dynamical system representing a controlled rigid spacecraft given by*

$$\dot{x}(t) = -I_{\rm b}^{-1} X I_{\rm b} x(t) + I_{\rm b}^{-1} u(t), \qquad x(0) = x_0,\ t \geq 0, \tag{58}$$

where $x = [x_1, x_2, x_3]^{\rm T}$ represents the angular velocities of the spacecraft with respect to the body-fixed frame, $I_{\rm b} \in \mathbb{R}^{3\times 3}$ is an unknown positive-definite inertia matrix of the spacecraft, $u(t) = [u_1, u_2, u_3]^{\rm T}$ is a control vector with control inputs providing body-fixed torques about three mutually perpendicular axes defining the body-fixed frame of the spacecraft, and X denotes the skew-symmetric matrix

$$X \triangleq \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \tag{59}$$

Note that (58) can be written in state space form (1) with $f(x) = -I_{\rm b}^{-1} X I_{\rm b} x$ and $G(x) = I_{\rm b}^{-1}$. Since $f(x)$ is a quadratic function, we parameterize $f(x)$ as $f(x) = \Theta_{n\ell} f_{n\ell}(x)$, where $\Theta_{n\ell} \in \mathbb{R}^{3\times 6}$ is an unknown matrix and $f_{n\ell}(x) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1]^{\rm T}$. Next, let $F(x) = [x^{\rm T}, f_{n\ell}^{\rm T}]^{\rm T}$, $B_{\rm r} = I_3$, $\hat{G}(x) \equiv I_3$, $\hat{K}_1 = I_{\rm b}$, and $\hat{K}_2 = [A_{\rm r}, -\Theta_{n\ell}]$, so that

$$\begin{aligned} G(x)\hat{G}(x)\hat{K}_1 &= I_{\rm b}^{-1} I_3 I_{\rm b} = I_3 = B_{\rm r}, \\ f(x) + B_{\rm r}\hat{K}_2 F(x) &= f(x) + I_3 \left[A_{\rm r}, -\Theta_{n\ell}\right] F(x) = A_{\rm r} x, \end{aligned}$$

and hence (3) and (4) hold. Now, it follows from Theorem 3.1 that the adaptive feedback controller (16) guarantees that $e(t) \to 0$ as $t \to \infty$ when considering input amplitude and rate saturation constraints. Specifically, here we choose $R_1 = 0.5 I_3$, $R_2 = 0.1 I_3$, and

$$A_{\rm r} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{bmatrix}.$$

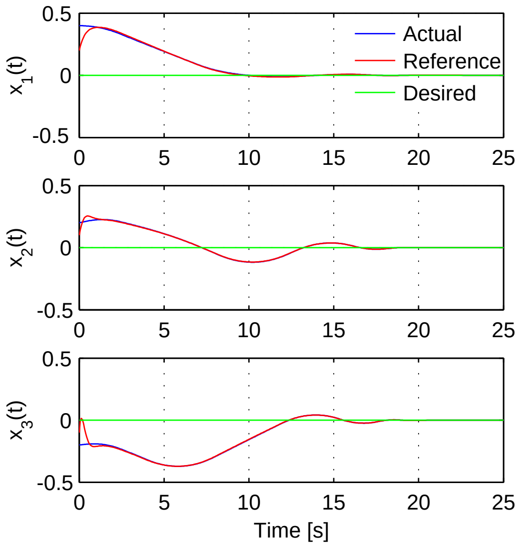

Figure 1. Angular velocities versus time

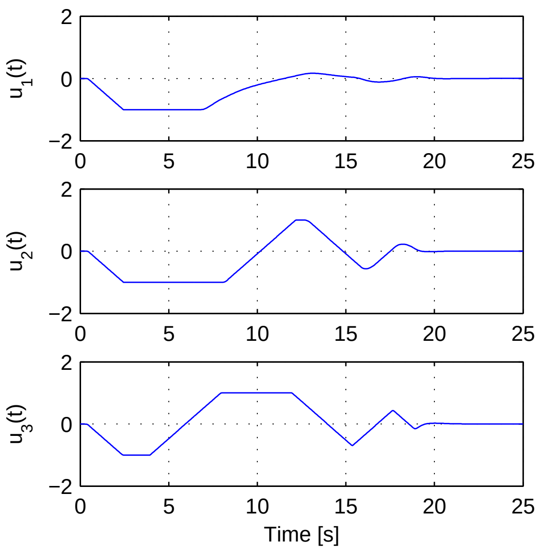

Figure 2. Control signals versus time

To analyze this design we assume that

$$I_{\rm b} = \begin{bmatrix} 20 & 0 & 0.9 \\ 0 & 17 & 0 \\ 0.9 & 0 & 15 \end{bmatrix}, \quad Q_1 = Q_2 = I_3,$$

with initial condition $x(0) = [0.4, 0.2, -0.2]^{\rm T}$, $x_1(0) = \frac{1}{2}x(0)$, $u(0) = x_{r2}(0) = [0,0,0]^{\rm T}$. *Figure 1 shows the angular velocities versus time, with saturation constraints* $u_{\max} = 1$ *and* $\dot{u}_{\max} = 0.5$. *These angular velocities converge to zero. Figure 2 shows the corresponding control inputs.*

6 CONCLUSION

A direct adaptive nonlinear tracking control framework for multivariable nonlinear uncertain systems with actuator amplitude and rate saturation constraints was developed. By appropriately modifying the adaptive control signal to the reference system dynamics, the proposed approach guarantees asymptotic stability of the error system dynamics in the face of actuator amplitude and rate limitation constraints. Finally, a numerical example was presented to show the utility of the proposed adaptive control scheme.

REFERENCES

[1] F. Ohkawa and Y. Yonezawa, “A discrete model reference adaptive control system for a plant with input amplitude constraints,” *Int. J. Contr.*, vol. 36, pp. 747–753, 1982.

[2] D. Y. Abramovitch, R. L. Kosut, and G. F. Franklin, “Adaptive control with saturating inputs,” *Proc. IEEE Conf. Dec. Contr.*, pp. 848–852, Athens, Greece, 1986.

[3] A. N. Payne, “Adaptive one-step-ahead control subject to an input-amplitude constraint,” *Int. J. Contr.*, vol. 43, pp. 1257–1269, 1986.

[4] C. Zhang and R. J. Evans, “Amplitude constranied adaptive control,” *Int. J. Contr.*, vol. 46, pp. 53–64, 1987.

[5] S. P. Kárason and A. M. Annaswamy, “Adaptive control in the presence of input constraints,” *IEEE Trans. Autom. Contr.*, vol. 39, pp. 2325–2330, 1994.

[6] A. M. Annaswamy and S. P. Kárason, “Discrete-time adaptive control in the presence of input constraints,” *Automatica*, vol. 31, pp. 1421–1431, 1995.

[7] K. McKay, “Summary of an AGARD workshop on pilot induced oscillation,” *Proc. AIAA Guid. Nav. Contr.*, pp. 1151–1161, 1994.

[8] R. A. Hess and S. A. Snell, “Flight control design with rate saturating actuators,” *AIAA J. Guid. Contr. Dyn.*, vol. 20, pp. 90–96, 1997.

[9] A. Leonessa, W. M. Haddad, and T. Hayakawa, “Adaptive control for nonlinear uncertain systems with actuator amplitude and rate saturation constraints,” *American Control Conference, Arlington, VA*, 2000.

[10] H. K. Khalil, *Nonlinear Systems*. Upper Saddle River, NJ: Prentice-Hall, 1996.

[11] M. Krstić, I. Kanellakopoullos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. New York, NY: John Wiley and Sons, Inc, 1995.

[12] D. Swaroop, J.-K. Hedrick, P.-P. Yip, and J.-C. Gerdes, “Dynamic surface control for a class of nonlinear systems,” *IEEE Trans. Autom. Contr.*, vol. 45, pp. 1893–1899, 2000.