

A Metric for Planar Displacements

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ABSTRACT

Metrics, which facilitate the measurement of parameters such as “distance” and “length” are used frequently in rigid body guidance problems. Commonly used metrics have a characteristic of being dependent on the choice of fixed or moving reference frame and the units used. Most motion synthesis algorithms require some notion of the “distance” between two desired locations¹. The metrics in Euclidean space depend on the coordinate frame and units used. A metric independent of these choices is desirable. In this paper we present a metric which is independent of the choice of fixed coordinate frame.

Keywords

Distance metric, polar decomposition, motion synthesis.

1. INTRODUCTION

The approximate synthesis of mechanisms for rigid body guidance can be carried out only when there is a means to measure the distance between the moving frame of the mechanism and the desired locations. A metric is used to measure the distance between two points in a set. There are various metrics for finding the distance between two points in Euclidean space. Finding the distance between two locations of a rigid body is still the subject of ongoing research, see [17, 2, 8, 12, 18, 15, 7, 10, 20, 3, 6]. For two locations of a finite rigid body either SE(2)(planar) or SE(3)(spatial) locations the metrics being used yield a distance which is dependant upon the chosen frame of reference and the units used, see [2, 18]. But, a metric independent of these choices is desirable and is referred to as bi-invariant. Metrics independent of the choice of coordinate frames and the units used do exist in SO(N), see Larochele [15]. One bi-invariant metric defined by Ravani and Roth [19], defines the orientation between two orientations of a rigid body as the magnitude of the difference between the associated quaternions. The techniques used here are based on the polar decomposition (represented by PD) of the homogenous transform representation of the elements of SE(N). The mapping of the elements of SE(N-1) to SO(N) yields hyperdimensional rotations that approximate the rigid body displacements. A conceptual representation of the mapping of SE(N-1) to SO(N) is shown in Figure 1. In the planar case the elements of SE(2) are mapped onto the

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¹Location of a rigid body prescribes both its position and orientation

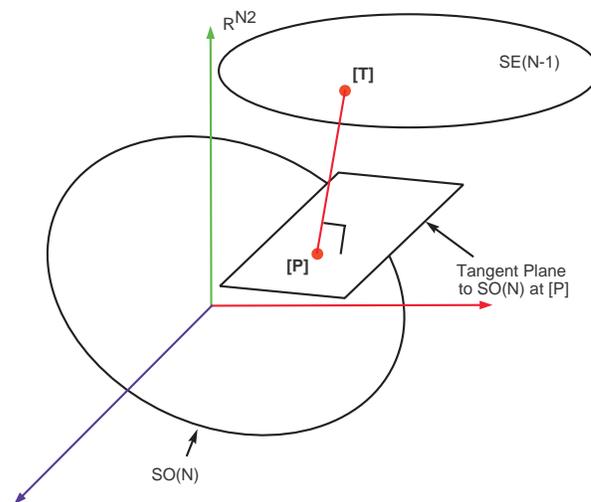


Figure 1: SE(N-1) to SO(N)

SO(3) as shown in Figure 2. Once the elements are mapped to SO(N) they can then be evaluated using the bi-invariant metrics existing in SO(N). The PD based projection metric in SE(N-1) is left invariant.

2. FINITE SETS OF LOCATIONS

Consider the case when a finite number of n displacements ($n \geq 2$) are given and we have to find out the magnitude of these displacements in order to synthesize a mechanism for rigid body guidance through the given locations. The displacements would then depend on the coordinate frame and the system of units chosen. In order to yield a left invariant metric we utilize a unit point mass model for a moving body suggested by Larochele, [14]. This is done to avoid complicated volume integrals over the body. The center of mass and the principal frame are unique for the mechanical system and invariant with respect to the choice of the coordinate frames and the system of units. The set of all two-dimensional rigid body displacements forms a group generally referred to as SE(2), the special Euclidean group in two dimensions.

The procedure for determining the center of mass \vec{c} and the principal axes frame (PF) associated with the n prescribed locations is described below. A unit point mass is located at the origin of each location as shown in Figure 3.

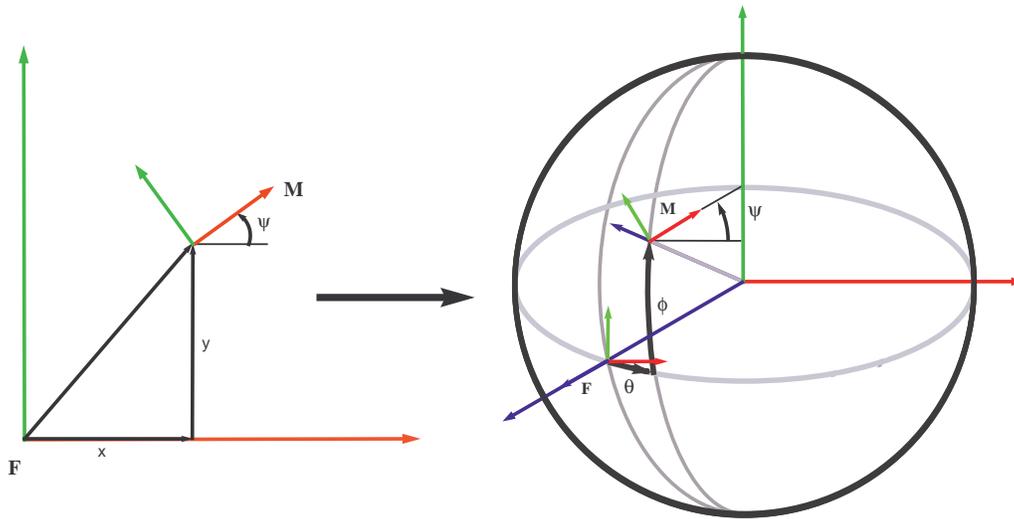


Figure 2: SE(2) to SO(3)

$$\vec{c} = \frac{1}{n} \sum_{i=1}^n \vec{d}_i \quad (1)$$

where, \vec{d}_i is the translation vector associated with the i^{th} location (i.e. the origin of the i^{th} location with respect to the principal frame).

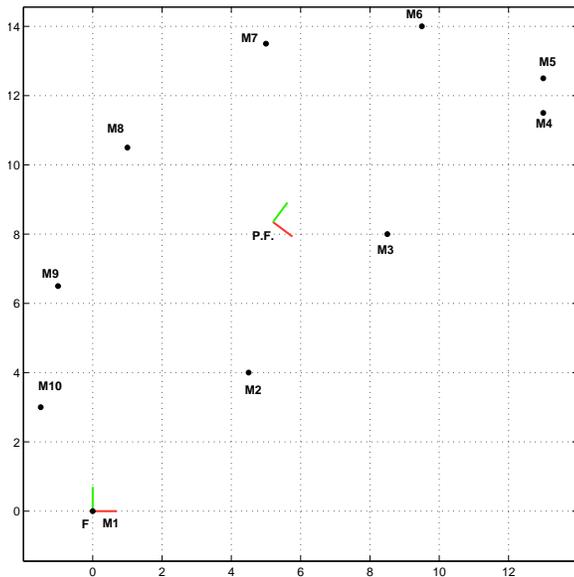


Figure 3: Unit Point Mass Model

In order to determine the principal frame which is defined as the principal axes of the n point mass system with its origin at the centroid \vec{c} . After finding the centroid of the mass system we determine the inertia tensor $[I]$ associated with the n point masses;

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad (2)$$

The principal moments of inertia shown above are defined by,

$$\begin{aligned} I_{xx} &= - \sum_{i=1}^n (y_i^2 + z_i^2) \\ I_{yy} &= - \sum_{i=1}^n (z_i^2 + x_i^2) \\ I_{zz} &= - \sum_{i=1}^n (x_i^2 + y_i^2) \end{aligned} \quad (3)$$

the products of inertia are,

$$\begin{aligned} I_{xy} &= I_{yx} = - \sum_{i=1}^n (x_i y_i) \\ I_{xz} &= I_{zx} = - \sum_{i=1}^n (x_i z_i) \\ I_{yz} &= I_{zy} = - \sum_{i=1}^n (y_i z_i) \end{aligned} \quad (4)$$

and x_i, y_i, z_i are the components of \vec{d}_i .

The principal frame is thus determined as,

$$[PF] = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{c} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (5)$$

where, \vec{v}_i are the principal axes associated with the inertia tensor, see Greenwood [9]. The directions of the vectors along the principal axes (\vec{v}_i) are chosen such that the principal frame is a right handed system.

In the planar case the inertia tensor [I] reduces to

$$[I] = \begin{bmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

and, the principal frame for the planar case reduces to a 3 × 3 matrix as shown:

$$[PF] = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{c} \\ 0 & 0 & 1 \end{bmatrix} \quad (7)$$

where, \vec{v}_1 and \vec{v}_2 are the lines representing the principal frame. The principal frame is independent of the orientations of the frames representing the desired locations. The metric however depends on the orientations of the frames.

3. DETERMINATION OF THE PRINCIPAL FRAME

A primary concern in formulating the design problem is the dependence of representations of SE(N-1) on the chosen coordinate frames. Any optimization procedure that minimizes a measure of distance between the task space and the workspace of a mechanism must necessarily depend on the choice of coordinate frames (fixed and moving). This means that an optimum design computed in one coordinate frame will, in general, differ from that computed in another frame. There are several strategies for avoiding this situation, see for example, Chirikjian [3]. The center of mass and the principal axes frame are unique for the mechanical system and invariant with respect to both the choice of fixed coordinate frames as well as the system of units [9, 1].

The principal frame is determined by using the inertia tensor [I]. The principal axes associated with the principal frame are chosen in such a way as to form a right-handed frame. In the planar case there are 4 possible orientations of the PF as can be seen from Figure 4. The directions of the principal frame are chosen in such a way as to align it as closely as possible with the fixed frame. One of the ways that this may be achieved is by computing the dot product of one of the axes of the principal frame with the fixed frame.

The four different right handed principal frames that are possible are given by,

$$\begin{aligned} \vec{v}_1 \times \vec{v}_2 &= [0 \ 0 \ 1]^T \\ \vec{v}_2 \times -\vec{v}_1 &= [0 \ 0 \ 1]^T \\ -\vec{v}_1 \times \vec{v}_2 &= [0 \ 0 \ 1]^T \\ -\vec{v}_2 \times \vec{v}_1 &= [0 \ 0 \ 1]^T \end{aligned} \quad (8)$$

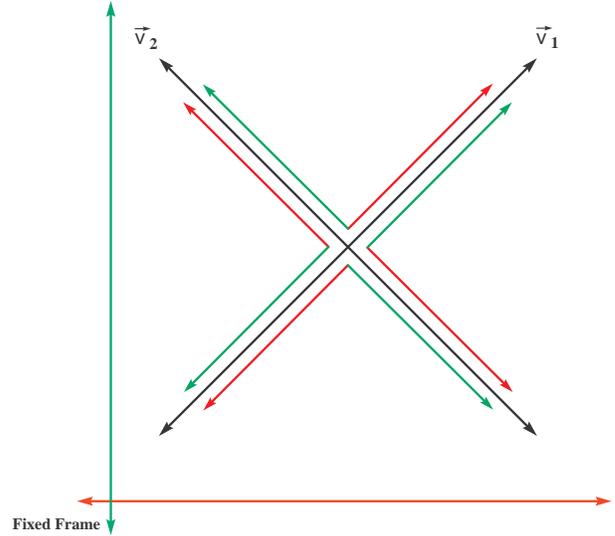


Figure 4: Four Possible Orientations for the PF

The angle θ_i formed by the four principal frames and the fixed frame can be found by,

$$\begin{aligned} \theta_1 &= \cos^{-1}(\vec{v}_1 \cdot [1 \ 0 \ 0]^T) \\ \theta_2 &= \cos^{-1}(\vec{v}_2 \cdot [1 \ 0 \ 0]^T) \\ \theta_3 &= \cos^{-1}(-\vec{v}_1 \cdot [1 \ 0 \ 0]^T) \\ \theta_4 &= \cos^{-1}(-\vec{v}_2 \cdot [1 \ 0 \ 0]^T) \end{aligned} \quad (9)$$

where, $[1 \ 0 \ 0]^T$ represents the unit vector in the positive direction of the X axis. The direction of the principal frame with the minimum positive value of θ_i is chosen.

4. COMPUTATION OF CHARACTERISTIC LENGTH

The unit disparity between translation and rotation is resolved by normalizing the translational terms in displacements. The displacements are normalized by choosing a characteristic length, R . Investigations on the use and determination of characteristic lengths appear in Larochele, [15, 4]. The characteristic length used, based upon the investigations reported in [7, 13], is $\frac{24L}{\pi}$, where L is the maximum translational component in the set of displacements at hand. This characteristic length is the radius of the hypersphere that approximates the translational terms by angular displacements that are ≤ 7.5 degrees. It was shown in [16] that this radius yields an effective balance between translational and rotational displacement terms for projection metrics. The PD metric is dependent on the choice of characteristic length. Larger characteristic length results in an increase in the weight on the rotational terms whereas a decrease in the characteristic length results in increase in weight on the translational terms.

5. DUBRULLE'S ALGORITHM

A number of iterative algorithms exist for the evaluation of the polar decomposition. Hingham described a method based upon Newtons method, see [11]. A simple and

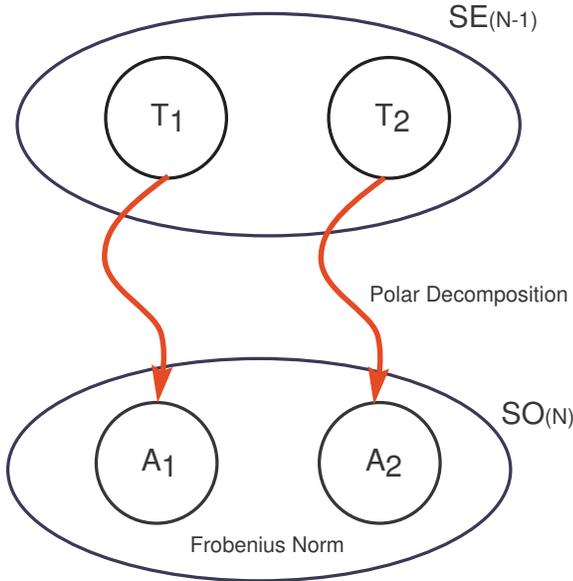


Figure 5: Mapping to SO(N)

efficient iterative algorithm for the computation of the polar decomposition is shown by Dubrulle [5]. The algorithm produces mono-tonic convergence in the Frobenius norm that delivers an IEEE solution in ~ 10 or fewer steps.

```

%% Dubrulle's Algorithm
%% Input T : Scaled transformation
%% matrix in SE(N-1)
%% Output P : Projection in SO(N)
function P = polarmetric(T)
%Initialization
P = T;
limit = (1+eps)* sqrt(size(T,2));
T = inv(P');
g = sqrt(norm(T,'fro')/norm(P,'fro'));
P = 0.5*(g*P+(1/g*T));
f = norm(P,'fro');
pf = inf;
% Iteration
while (f>limit) & (f<pf)
    pf = f;
    T = inv(P');
    g = sqrt(norm(T,'fro')/f);
    P = 0.5*(g*P+(1/g*T));
    f = norm(P,'fro');
end
return

```

6. DISTANCE BETWEEN ELEMENTS IN SO(N)

The elements in SO(N) are derived from homogenous transformations representing planar SE(2) or spatial SE(3) displacements by polar decomposition as shown in Figure 5. Planar and spatial displacements may then be approximated using a 3×3 rotation in the planar case or a 4×4 rotation in the spatial case. The elements T_i in the planar case are

given by

$$T_i = \begin{bmatrix} [R] & | & \vec{t} \\ \hline 0 & 0 & | & 1 \end{bmatrix} \quad (10)$$

where, $[R]$ is a 2×2 matrix representing the rotational component and \vec{t} represents the translational component of the homogenous transformation of the planar locations. The scaled transformation matrices are thus obtained as,

$$T_{i(scaled)} = \begin{bmatrix} [R] & | & \vec{t}/R \\ \hline 0 & 0 & | & 1 \end{bmatrix} \quad (11)$$

where, R represents the characteristic length described in Section 4. The scaled transformation matrices are then projected to SO(3) by using the Dubrulle algorithm for polar decomposition. The distance between elements in SO(3) can be determined by using the metric suggested by Larochele, [14]. The distance between two elements $[A_1]$ and $[A_2]$ in SO(N) can be defined by using the Frobenius norm as follows,

$$d = \|[I] - [A_2][A_1]^T\|_F \quad (12)$$

7. SUMMARY OF THE TECHNIQUE

For a set of n finite locations the steps to be followed are:

1. Determine the PF of the n locations.
2. Determine the relative displacements from PF to each of the n locations.
3. Determine the characteristic length R associated with the n displacements with respect to the PF and scale the translation terms in each by $1/R$.
4. Compute the projections of PF and each of the scaled relative displacements using the polar decomposition algorithm explained in Section 5.
5. The magnitude of the displacement is defined as the distance from PF to the scaled relative displacement as computed via Equation 12. The distance between any two of the n locations is similarly computed by the application of Equation 12 to the projected scaled relative displacements.

It is to be noted that even though the principal frame does not depend on the orientations of the desired locations the metric does.

8. CASE STUDY

Consider the rigid body guidance problem investigated by Larochele [13]. The 10 planar locations are listed in Table 1 and the origins of the coordinate frames with the respect to the fixed reference frame (F) are shown in Figure 3.

Table 1: 10 Rigid Body Locations

No.	x	y	α
1	0.0	0.0	40.0
2	4.5	4.0	20.0
3	8.5	8.0	0.0
4	13.0	11.5	-30.0
5	13.0	12.5	-35.0
6	9.5	14.0	-35.0
7	5.0	13.5	-30.0
8	1.0	10.5	-15.0
9	-1.0	6.5	0.0
10	-1.5	3.0	20.0

Table 2: Distances to the Desired Locations

No.	x	y	α	d
1	0.0	0.0	40.0	1.7280
2	4.5	4.0	20.0	1.3115
3	8.5	8.0	0.0	0.8567
4	13.0	11.5	-30.0	0.1584
5	13.0	12.5	-35.0	0.1487
6	9.5	14.0	-35.0	0.0920
7	5.0	13.5	-30.0	0.1516
8	1.0	10.5	-15.0	0.5017
9	-1.0	6.5	0.0	0.8589
10	-1.5	3.0	20.0	1.3134

The principal frame was determined using the methodology detailed in section 3,

$$[PF] = \begin{bmatrix} 0.80 & 0.60 & 5.20 \\ -0.60 & 0.80 & 8.35 \\ 0 & 0 & 1 \end{bmatrix} \quad (13)$$

and is shown in Figure 3. Consider an approximate motion synthesis problem in which a planar four-bar is desired to attain these locations. An algorithm to perform the approximate motion synthesis would require a distance metric to enable the measurement of the distance from the moving frame attached to the coupler, to each of the desired locations. Here, we present an example in which the mechanisms coupler frame is nearest the sixth location. In Table 2 the distances from the coupler frame to each of the 10 desired locations are listed. Figure 6 shows the planar 4R chain when it is nearest the sixth location.

9. CONCLUSIONS

This paper presents a polar decomposition based distance metric as it applies to planar locations. A method for determination of the invariant frame for a finite set of displacements viz: the principal frame has been developed. The homogenous transform representations of elements in SE(2) are then projected onto SO(3) by using polar decomposition. The polar decomposition of elements of SE(2) yields an element of SO(3) nearest to the given element of SE(2). A bi-invariant metric is then utilized to find the distance between elements in SO(3). The result is a left invariant metric on planar displacements. This metric is used to find the distances between rigid body locations.

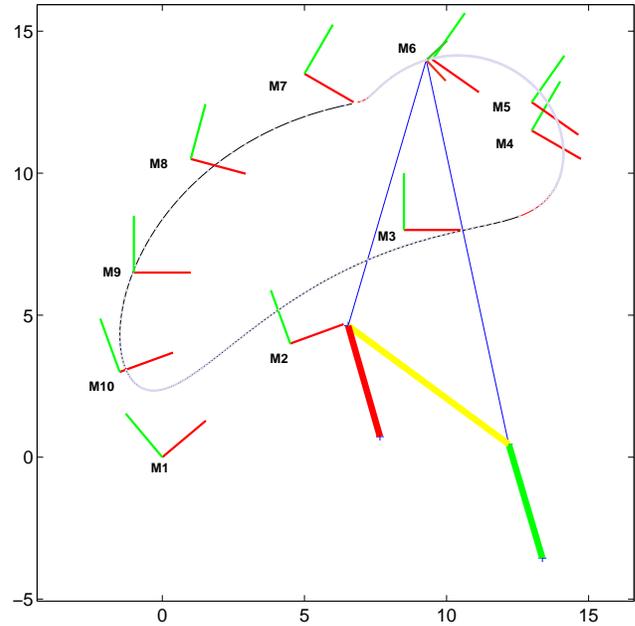


Figure 6: A Planar 4R Chain nearest the Sixth Location

10. ACKNOWLEDGMENTS

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