Evaluating the Fault Tolerance of a Parallel Manipulator Based on Relative Manipulability Indices

Hyun Geun Yu and Rodney G. Roberts
Department of Electrical and Computer Engineering
FAMU-FSU College of Engineering
Florida State University
Tallahassee, FL 32310-6046
Email: hyun@eng.fsu.edu, rroberts@eng.fsu.edu

ABSTRACT

In this article, the authors investigate the fault tolerance of manipulators in their nominal configuration. In this work, fault tolerance is measured in terms of the worst case relative manipulability index. While this approach is applicable to both serial and parallel mechanisms, it is especially applicable to parallel mechanisms with a limited workspace. It is first shown that the relative manipulability indices are characterized by the null space of the manipulator Jacobian. This motivates the problem of determining the class of manipulator Jacobians with a prescribed null space. This approach can be used to find optimally fault-tolerant manipulators. It is then shown through dimensional arguments that there are limits to the amount of redundancy for this problem to be solvable. The authors use these limits to prove that a previously derived inequality for the worst case relative manipulability index is generally not achieved for fully spatial manipulators and that the concept of optimal fault tolerance to multiple failures is more subtle than previously indicated. After presenting an example of a seven degree-of-freedom mechanism that is optimally fault-tolerant to single failure, the authors consider the problem of finding a manipulator Jacobian that is optimally fault tolerant to multiple failures. It is shown that optimal solutions cannot be equally fault tolerant.

Keywords

kinematic redundancy, fault tolerance, manipulability, parallel manipulators

1. INTRODUCTION

Fault tolerant design of serial or parallel manipulators is critical for tasks requiring robots to operate in remote and hazardous environments where repair and maintenance tasks are extremely difficult [1]-[5]. In such cases, operational reliability is of prime importance. By adding kinematic redundancy to the robotic system, the robot may still be able to perform its task even if one or more joint actuators fail [6]. However, simply adding kinematic redundancy to the system does not guarantee fault tolerance [7]. One must strategically plan how the kinematic redundancy should be added to the system to ensure that fault tolerance is optimized [8].

One approach to the problem of designing fault tolerant robots is to optimize some measure of fault tolerance. There are a variety of kinematic measures proposed [9]-[13]. One particular measure is the *manipulability index* [14]:

$$w(J) = \sqrt{\det(JJ^T)}. \tag{1}$$

where J is the manipulator Jacobian of the robot. The manipulability index is a nonnegative quantity that takes on the value zero precisely at the singular configurations of the robot. Configurations that result in a relatively large manipulability index are usually considered to be good operating configurations.

Zhang, Duffy, and Crane defined the *quality index* to quantify the performance of a spatial redundant in-parallel manipulator [15]-[17]:

$$\lambda = \sqrt{\frac{\det(JJ^T)}{\det(J_m J_m^T)}} \tag{2}$$

where J is the six-by-eight manipulator Jacobian at the current configuration and J_m is the manipulator Jacobian at the central symmetrical configuration. The quality index is a dimensionless ratio which takes a maximum value of 1 at a central symmetrical configuration that is shown to correspond to the maximum value of the square root of the determinant of the product of the manipulator Jacobian with its transpose. An important property of the quality index is that it avoids some of the dimensional inconsistencies associated with the manipulability index.

In this article we focus on the relative manipulability index, which was first introduced in [7] to quantify the fault tolerance of kinematically redundant serial manipulators. Let J be an $m \times n$ Jacobian where $m < n$ and suppose that there are $f \leq n-m$ joints that are locked. The *relative manipulability index* corresponding to locked joint failures in joints $i_1, \ldots, i_f$ is defined to be

$$\rho_{i_1,\cdots,i_f} = \frac{w({}^{i_1\cdots i_f}J)}{w(J)} \tag{3}$$

where J denotes the manipulator Jacobian, ${}^{i_1\cdots i_f}J$ denotes the manipulator Jacobian after the columns $i_1, \ldots, i_f$ corresponding to the failed joints are removed, and where $w(J) = \sqrt{\det(JJ^T)}$ is the manipulability index for J [14]. For a revolute serial manipulator or a parallel mechanism, the relative manipulability index, like the quality index, avoids the dimensional inconsistencies inherent in the manipulability index. The relative manipulability index is a local measure of the amount of dexterity that is retained when a manipulator suffers one or more locked joint failures. The value of a relative manipulability index ranges from zero to one. A zero value would indicate a loss of full end-effector motion at that configuration after the failed joints are locked. In other words, a zero relative manipulability index means that the reduced manipulator Jacobian ${}^{i_1\cdots i_f}J$ does not have full rank. A relative manipulability index of one would indicate that no dexterity is lost at that configuration. In this case the joints in question do not contribute to end-effector motion at the operating configuration prior to their failure, i.e., those joints only produce self-motion [7].

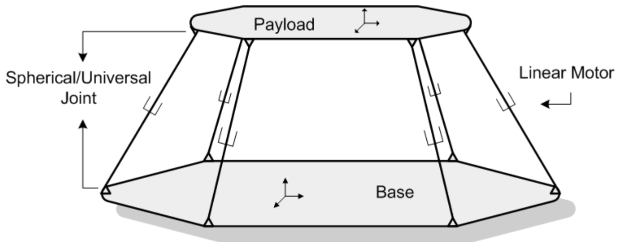

Fig. 1. A General Gough-Stewart Platform (GSP)

Relative manipulability indices have also been used to study the fault tolerance of redundant Gough-Stewart platforms [18]. A Gough-Stewart platform (GSP) is a parallel mechanism consisting of a base, a moving platform, and struts as shown in Fig 1. For a GSP, the inverse Jacobian M maps the generalized velocity of the payload to the corresponding joint velocities of the individual struts. The matrix M has the same form as the transpose of a manipulator Jacobian J. In other words, the first three components of each row forms a unit vector that is orthogonal to the vector given by the last three components of that row. If M^TM is a diagonal matrix, then one says that the mechanism is an orthogonal Gough-Stewart platform (OGSP) [19], [20]. OGSPs are a special class of GSPs that are particularly well-suited to various precision applications owing to the local kinematic and dynamic decoupling of the Cartesian directions they provide [21]. In [18], a class of OGSPs was identified that possess optimal fault tolerant manipulability for single joint failures based on maximizing the minimum relative manipulability index about an operating point.

In this article, the authors investigate the fault tolerance of manipulators at their nominal operating configuration when there are single or multiple locked joint failures. In the next section, the relationship between the relative manipulability indices and the null space of the manipulator Jacobian is established using the principal minors of the null space projection operator. Based on this formulation of fault tolerance, it is easy to establish identities and inequalities for the relative manipulability indices. Motivated by the observation that the relative manipulability indices are completely determined by the null space of the manipulator Jacobian, we then discuss some of the theoretical limitations of designing manipulator Jacobians with a prescribed null space. An optimally fault tolerant seven degree-of-freedom (DOF) manipulator is then determined in Section 3. In Section 4, the authors consider the concept of equally fault tolerant configurations, i.e., configurations for which any combination of a specified number of joint failures results in the same local manipulability. It is shown through a series of results that such configurations are truly rare. Conclusions and future work appear in Section 5.

2. FAULT TOLERANCE AND THE NULL SPACE OF THE MANIPULATOR JACOBIAN

It turns out that the amount of fault tolerance that a manipulator possesses is closely related to the null space of the manipulator Jacobian. This important fact motivates the problem of designing operating configurations for robotic mechanisms based on choosing the manipulator Jacobian to have a prescribed null space. After characterizing the relative manipulability indices in terms of the null space of the manipulator Jacobian, we will discuss the amount of freedom that a designer has in choosing the null space of a nominal manipulator Jacobian.

2.1 Relative Manipulability Indices and the Null Space of the Manipulator Jacobian

We begin by demonstrating that the subdeterminants of the null space projection operator of the manipulator Jacobian completely characterize the relative manipulability indices. Our analysis is applicable to serial and parallel mechanisms so throughout this work we will use M and J^T interchangeably. Let J be a full rank $m \times n$ matrix with $m < n$ and let $r = n - m$. For a manipulator, m denotes the dimension of the workspace, n denotes the number of joints, and r denotes the degree of redundancy. We will call an $n \times r$ matrix N a *null space matrix* of J if the columns of N form an orthonormal basis for the null space of J. Although the null space matrix N is not unique for a given J, any two null space matrices N and N' of J are related by an orthogonal matrix Q in the following way: $N' = NQ$. We will see later that we can use Q to place N into a canonical form that can help us to properly view the null space and its relationship to fault tolerance.

In [7], it was shown that the relative manipulability index is related to the null space matrix by the relationship

$$\rho_{i_1,\cdots,i_f} = w(N_{i_1\cdots i_f}) = \sqrt{|N_{i_1\cdots i_f} N^T_{i_1\cdots i_f}|} \tag{4}$$

where $N_{i_1\cdots i_f}$ is the $f \times r$ matrix consisting of rows $i_1,\dots,i_f$ of the matrix N. We thus have the interesting observation that the relative manipulability indices are strictly a function of the null space of J. We will build on this result to address the issue of designing manipulators that are optimally fault tolerant to one or more joint failures.

The relative manipulability index squared, $\rho^2_{i_1,\cdots,i_f} = |N_{i_1\cdots i_f} N^T_{i_1\cdots i_f}|$, is perhaps best viewed as a principal minor of the null space projection operator $P_N = I - J^+J$ where J^+ denotes the pseudoinverse of J. The $n \times n$ matrix P_N represents the orthogonal projection of the joint space onto the null space of J. Unlike a null space matrix, P_N is unique for a given J; however, given a corresponding null space matrix N, we have that $P_N = NN^T$. It then follows from (4) that the relative manipulability index squared is equal to the determinant of the matrix consisting of the $i_1,\dots,i_f$ rows and columns of P_N.

Recall that an $k \times k$ *minor* of an $n \times n$ matrix $A = [a_{ij}]$ with $k < n$ is a subdeterminant of the form

$$A\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix} \triangleq \begin{vmatrix} a_{i_1j_1} & a_{i_1j_2} & \cdots & a_{i_1j_k} \\ a_{i_2j_1} & a_{i_2j_2} & \cdots & a_{i_2j_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_kj_1} & a_{i_kj_2} & \cdots & a_{i_kj_k} \end{vmatrix} \tag{5}$$

where $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$. If $(j_1,\dots,j_k) = (i_1,\dots,i_k)$, then this quantity is called a *principal minor* of A. Hence, we have that $\rho^2_{i_1,\cdots,i_f}$ is the $(i_1,\dots,i_f)$ principal minor of $P_N = NN^T$:

$$\rho^2_{i_1,\cdots,i_f} = P_N\begin{pmatrix} i_1 & \cdots & i_f \\ i_1 & \cdots & i_f \end{pmatrix}. \tag{6}$$

It is well known that the coefficients of the characteristic polynomial $p_A(\lambda) = |\lambda I - A|$ of A are given in terms of the sums of the principal minors of A. To be more specific, for $p_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$, we have that

$$a_{n-k} = (-1)^k \sum_{1 \le i_1 < \cdots < i_k \le n} A\begin{pmatrix} i_1 \cdots i_k \\ i_1 \cdots i_k \end{pmatrix}. \tag{7}$$

Since P_N is a projection, it is idempotent, i.e., $P_N^2 = P_N$, so its only possible distinct eigenvalues are 0 and 1. Furthermore, because $\mathrm{rank}(P_N) = r < n$ where $r = n - m$, it follows that the characteristic polynomial of P_N is

$$p(\lambda) = \lambda^m(\lambda - 1)^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \lambda^{n-k}. \tag{8}$$

Equations (6), (7), and (8) then imply that

$$\sum_{1 \le i_1 < \cdots < i_f \le n} \rho^2_{i_1,\cdots,i_f} = \binom{r}{f}. \tag{9}$$

This result, written as a slightly different but equivalent expression, was also proven in [18]; however, the proof provided there was based on repeated application of the Binet-Cauchy theorem and was less direct than applying principal minors. It is important to note, however, that the approach just given is not merely a different proof of the result in [18]. More importantly, it provides us with an approach that will be used in Section 4 to address multiple joint failures.

As noted in [18], equation (9) can be used to obtain an upper bound for the worst case relative manipulability index by noting that the minimum value of any set of numbers must be less than or equal to the average so that

$$\min_{1 \le i_1 < \cdots < i_f \le n} \rho_{i_1,\cdots,i_f} \le \sqrt{\frac{\binom{r}{f}}{\binom{n}{f}}}. \tag{10}$$

This inequality provides us with some insight into the question of how fault tolerant a manipulator can be.

2.2 Designing Nominal Fully Spatial Manipulator Jacobians with a Prescribed Null Space

Based on the inequality in (10), Ukidve, et al., [18] convincingly argue the importance of designing for fault tolerance. This is especially true when there may be multiple faults. One approach to ensuring local fault tolerance is to design the manipulator based on null space properties. This is particularly applicable when the required workspace is very small as is the case in [18]. However, there are limitations to how much redundancy can be used when designing nominal manipulator Jacobians with a prescribed null space.

These limitations follow from the fact that the manipulator Jacobian for a fully spatial manipulator must satisfy certain constraints on its columns. In particular, the vector given by the first three components of a column must have unit length and must be orthogonal to the vector given by the last three components of that column. For a manipulator with n joints, this results in $2n$ constraints. If the manipulator Jacobian is required to have a prescribed null space matrix, then each of its six rows must be orthogonal to the r rows of N^T where $r = n - 6$ is the number of degrees of redundancy of the manipulator. Consequently, the manipulator Jacobian must satisfy $6r$ null space constraints. Since the manipulator Jacobian has $6n$ parameters, it follows that one has $6n - 2n - 6r = 4(6+r) - 6r = 24 - 2r$ degrees of freedom to satisfy the design constraints. Hence, one cannot expect to arbitrarily find a manipulator with $r > 12$ degrees of redundancy that has a configuration where the manipulator Jacobian has a prescribed null space matrix.

If the mechanism is required to be an orthogonal Gough-Stewart platform (OGSP), then there is a further reduction in the degrees of freedom that one has in choosing a manipulator Jacobian with a prescribed null space. If JJ^T is required to be a diagonal matrix, there would be 15 additional constraints, decreasing the degrees of freedom to $9 - 2r$. In this case, one should not expect to be able to arbitrarily specify the null space of a manipulator with $r > 4$ degrees of redundancy. Of course there are cases where this is possible for the right choice of the null space. Furthermore, there could be cases where there is no real solution to the problem even though r is sufficiently small. The dimension arguments presented here do however provide the designer with a tool to assess the likely feasibility of designing a mechanism with prescribed null space properties and will be exploited in Section 4 to study the likely utility of a newly proposed fault tolerance concept.

TABLE I
PARAMETERS FOR THE OPTIMALLY FAULT TOLERANT 6×7 MANIPULATOR JACOBIAN GIVEN IN (11)

i	$\mathbf{n}_i^T$	$\mathbf{r}_i^T$
1	$[\ 0.000\ \ 0.000\ \ 1.000]$	$[-1.065\ \ 0.113\ \ 0.000]$
2	$[-0.172\ \ -0.827\ \ -0.536]$	$[\ 0.272\ \ -0.520\ \ 0.715]$
3	$[\ 0.877\ \ 0.418\ \ -0.239]$	$[\ 0.302\ \ -0.785\ \ -0.263]$
4	$[-0.408\ \ -0.004\ \ -0.913]$	$[\ 0.530\ \ 0.761\ \ -0.240]$
5	$[\ 0.473\ \ -0.802\ \ 0.364]$	$[\ 0.460\ \ -0.098\ \ -0.814]$
6	$[\ 0.065\ \ 0.983\ \ -0.174]$	$[-1.007\ \ 0.062\ \ -0.031]$
7	$[-0.836\ \ 0.233\ \ 0.497]$	$[\ 0.507\ \ 0.467\ \ 0.633]$

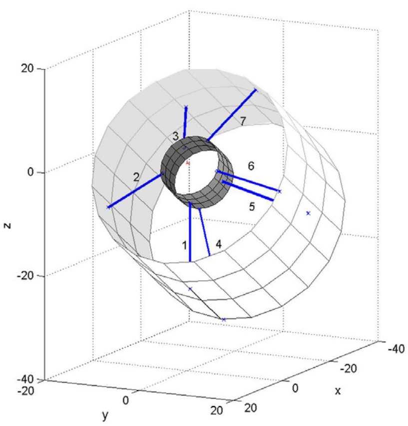

Fig. 2. An example of a cylindrical geometry for an OGSP corresponding to a realization of the optimally fault tolerant 6×7 manipulator Jacobian given in (11). The labels on the struts correspond to the respective columns of J (rows of M). Similar parallel mechanisms have been proposed for mounting in aerospace vehicles [19]. The $\mathbf{n}_i$'s for struts 2, 4, 6, and 7 have been reversed, i.e., the corresponding rows of (11) have been multiplied by -1.

3. DESIGNING OPTIMALLY FAULT TOLERANT 7-DOF SPATIAL MANIPULATOR JACOBIANS

According to equation (10), the maximum worst case relative manipulability index for a 7-DOF manipulator is $1/\sqrt{7}$. This optimal value is achieved if and only if the null vector of the manipulator Jacobian has components of equal magnitude, i.e., $|\hat{n}_i| = 1/\sqrt{7}$ where $\hat{n}_i$ is the i-th component of the unit length null vector $\hat{\mathbf{n}}_J$. Hence, we can specify the null vector to obtain an optimally fault tolerant manipulator configuration. Based on the dimension arguments in Section 2.2, we have 22 degrees of freedom in choosing a 7-DOF manipulator Jacobian with a prescribed null vector. If we further require that JJ^T be diagonal, the number of degrees of freedom in choosing J with a prescribed null vector reduces to seven. An example of a nominal manipulator

Jacobian that is optimally fault tolerant to a single failure is given by

$$J^T = \begin{bmatrix} 0.000 & 0.000 & 1.000 & 0.113 & 1.065 & 0.000 \\ -0.175 & -0.827 & -0.536 & 0.870 & 0.023 & -0.314 \\ 0.877 & 0.418 & -0.239 & 0.297 & -0.159 & 0.814 \\ -0.408 & -0.004 & -0.913 & -0.696 & 0.581 & 0.308 \\ 0.473 & -0.802 & 0.364 & -0.689 & -0.553 & -0.323 \\ 0.065 & 0.983 & -0.174 & 0.020 & -0.177 & -0.993 \\ -0.836 & 0.233 & 0.497 & 0.085 & -0.781 & 0.508 \end{bmatrix}. \tag{11}$$

This manipulator Jacobian corresponds to a 7-DOF manipulator, and its null vector components are all equal. Consequently, all seven relative manipulability indices corresponding to (11) are equal to $1/\sqrt{7}$. In this case, JJ^T is diagonal so (11) corresponds to an OGSP. The parallel mechanism parameters for the corresponding manipulator Jacobian are given by Table I. For a parallel manipulator, the unit vector $\mathbf{n}_i$ in the table indicates the direction of the i-th strut while $\mathbf{r}_i$ represents the point on the axis of the i-th strut that is closest to the origin.

There are a number of different possible manipulator realizations that can be generated from the Jacobian in (11). Clearly, the desired failure tolerance properties are not affected by multiplying one or more of the columns of J by -1. A parallel manipulator generated from this Jacobian is shown in Fig. 2.

4. EQUALLY FAULT TOLERANT CONFIGURATIONS

Equation (10) served as a motivation in [18] for defining a manipulator operating about a single point in the workspace to be optimally fault tolerant to $f \leq r$ failures if all of its relative manipulability indices $\rho_{i_1,\cdots,i_f}$ are equal, i.e.,

$$\rho_{i_1,\cdots,i_f} = \sqrt{\frac{\binom{r}{f}}{\binom{n}{f}}} \tag{12}$$

for $1 \leq i_1 < \cdots < i_f \leq n$. In this article, we will prefer to say that a manipulator is *equally fault tolerant to* $f \leq r$ *failures* at an operating configuration if (12) holds for $1 \leq i_1 < \cdots < i_f \leq n$ at that configuration. Note that equal fault tolerance is a local property since it would apply to specific configurations and would be most applicable for manipulators operating in a small workspace. If a manipulator is equally fault tolerant to $f \leq r$ failures, then by (10) it is optimally fault tolerant in a worst case relative manipulability index sense to $f \leq r$ failures. However, while it is clear that an optimal value exists, it is possible that a manipulator may not have a configuration that is equally fault tolerant to f failures. In this case, the optimal value is smaller than the bound given in (10). It is the goal of this section to show that this is typically the case.

Our first result concerning equally fault tolerant configurations is the following:

Theorem 1: If a manipulator is equally fault tolerant to f failures where $1 < f \leq r$, then it is also equally fault tolerant to $f-1$ failures. Furthermore, the manipulator is equally fault tolerant to k failures for $k = 1, 2, \ldots, f$.

Proof: We can prove the result by demonstrating that $\rho^2_{i_1,\cdots,i_{f-1}} = \binom{r}{f-1}/\binom{n}{f-1}$ for any $1 \leq i_1 < \cdots < i_{f-1} \leq n$. Rearranging the columns of N^T does not affect the overall fault tolerance analysis so we can assume without loss of generality that $i_1 = 1, \cdots, i_{f-1} = f-1$. Likewise, pre-multiplying N^T by an $r \times r$ orthogonal matrix Q does not affect the fault tolerance analysis. Hence, by applying a QR factorization, we can further assume without loss of generality that N^T has the form

$$N^T = \begin{bmatrix} N_{11} & N_{12} & N_{13} & \cdots & N_{1,f-1} & N_{1f} & N_{1,f+1} & \cdots & N_{1n} \\ 0 & N_{22} & N_{23} & \cdots & N_{2,f-1} & N_{2f} & N_{2,f+1} & \cdots & N_{2n} \\ 0 & 0 & N_{33} & \cdots & N_{3,f-1} & N_{3f} & N_{3,f+1} & \cdots & N_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{f-1,f-1} & N_{f-1,f} & N_{f-1,f+1} & \cdots & N_{f-1,n} \\ 0 & 0 & 0 & \cdots & 0 & N_{f,f} & N_{f,f+1} & \cdots & N_{f,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & N_{f+1,f+1} & \cdots & N_{f+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & N_{r,f+1} & \cdots & N_{r,n} \end{bmatrix}. \tag{13}$$

Now,

$$\rho^2_{1,\cdots,f} = |N_{11}N_{22}\cdots N_{f-1,f-1}N_{ff}|^2 = \frac{\binom{r}{f}}{\binom{n}{f}} \tag{14}$$

where the first equality in (14) follows by direct calculation from (14) and the second equality follows from the assumption that the manipulator is equally fault tolerant to f failures. Since pre-multiplying N^T by an orthogonal matrix does not affect the values of the relative manipulability indices, we can easily determine $\rho^2_{1,\ldots,f-1,j}$ for $j = f, \ldots, n$ by first pre-multiplying (14) by $\mathrm{diag}(I_{f-1}, U_j)$ where I_{f-1} is an $(f-1) \times (f-1)$ identity matrix and where U_j is an $(r-f+1) \times (r-f+1)$ orthogonal matrix that zeros out the last $r-f$ elements of the j-th column of N^T so that the (f,j) component of N^T becomes $\pm\alpha_j$ where $\alpha_j = \sqrt{N^2_{fj} + \cdots + N^2_{rj}}$. We then have that for $j = f, \ldots, r$,

$$\rho^2_{1,\ldots,f-1,j} = |N_{11}N_{22}\cdots N_{f-1,f-1}|^2\alpha_j^2. \tag{15}$$

Equating (15) to the second term in (14), we conclude that $\alpha_j = |N_{ff}|$ for $j = f, \ldots, n$ and zero otherwise so that $\sum_j \alpha_j^2 = (n-f+1)|N_{ff}|^2$. Now, the quantity $\sum_j \alpha_j^2$ is equal to the sum of the squares of the components of the last $r-f+1$ rows of N^T and since the rows of N^T have unit length, we have that $\sum_j \alpha_j^2 = r-f+1$ so that

$$|N_{ff}|^2 = \frac{r-f+1}{n-f+1}. \tag{16}$$

It then follows that

$$\rho^2_{1,\ldots,f-1} = \frac{\rho^2_{1,\ldots,f}}{|N_{ff}|^2} = \frac{\binom{r}{f}}{\binom{n}{f}}\,\frac{n-f+1}{r-f+1} = \frac{\binom{r}{f-1}}{\binom{n}{f-1}}. \tag{17}$$

Since the order of the columns did not matter, we conclude that the relative manipulability index for any $f-1$ failures is given by (17). Repeated application of this result implies that the manipulator is equally fault tolerant to k failures for $k = 1, 2, \ldots, f$. □

The reason that Theorem 1 will play such an important role in this regard is the fact that it forces P_N to have a particularly simple structure when the manipulator is equally fault tolerant to more than one failure. If J is equally fault tolerant to a single failure, then the diagonal elements of P_N are all equal to r/n. If J is equally fault tolerant to $f \geq 2$, then by Theorem 1 it is equally fault tolerant to single failures and to two failures. Hence, the (i,j) principal minor of the symmetric matrix P_N is

$$\begin{vmatrix} r/n & p_{ij} \\ p_{ji} & r/n \end{vmatrix} = \frac{r^2}{n^2} - p_{ij}^2 = \frac{r(r-1)}{n(n-1)} \tag{18}$$

where we have used the fact that $p_{ji} = p_{ij}$ and where the last equality follows from the assumption of equal fault tolerance to two failures. Solving for p_{ij} gives $p_{ij} = \frac{\pm 1}{n}\sqrt{\frac{r(n-r)}{n-1}}$ for all $1 \leq i < j \leq n$. Hence, when J is equally fault tolerant to $f \geq 2$

failures, the diagonal elements of P_N are all equal and the off-diagonal elements of P_N all have the same magnitude, i.e., P_N has the form

$$P_N = \begin{bmatrix} a & \pm b & \pm b & \cdots & \pm b \\ \pm b & a & \pm b & \cdots & \pm b \\ \pm b & \pm b & a & \cdots & \pm b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pm b & \pm b & \pm b & \cdots & a \end{bmatrix} \tag{19}$$

where $a = \frac{r}{n}$ and $b = \frac{-1}{n}\sqrt{\frac{r(n-r)}{n-1}}$.

Once again consider a manipulator with two degrees of redundancy, and suppose that the manipulator is equally fault tolerant to two failures. Since the rank of P_N would then be two, it follows that the 3×3 principal minors of P_N are zero; otherwise, the rank of P_N would be greater than or equal to three. Any 3×3 principal minor of P_N necessarily has the form

$$\begin{vmatrix} a & \pm b & \pm b \\ \pm b & a & \pm b \\ \pm b & \pm b & a \end{vmatrix} = a^3 - 3ab^2 \pm 2b^3. \tag{20}$$

Since one of these two quantities is zero, so is their product so that

$$\begin{aligned} 0 &= (a^3 - 3ab^2 + 2b^3)(a^3 - 3ab^2 - 2b^3) \\ &= (a-b)^2(a+2b)(a+b)^2(a-2b) \\ &= (a^2 - b^2)^2(a^2 - 4b^2). \end{aligned} \tag{21}$$

We thus conclude that $a^2 = b^2$ or $a^2 = 4b^2$. Substituting in the expressions for a and b yields that $n = 0$ or $n = 3$, respectively. As $n = 0$ does not make sense, we conclude that $n = 3$. Equivalently, the workspace has $m = n - r = 3 - 2 = 1$ degree of freedom so that the corresponding Jacobian is a 1×3 matrix. Equal fault tolerance then dictates that the Jacobian has the form $J = \begin{bmatrix} \pm\alpha & \pm\alpha & \pm\alpha \end{bmatrix}$ for some $\alpha > 0$.

The above observations prove the following result:

Theorem 2: No 8-DOF spatial manipulator can be equally fault tolerant to two simultaneous joint failures.

We are now ready to consider the case when J is equally fault tolerant to $f \geq 3$ failures. Applying similar arguments as above, we obtain the following result:

Theorem 3: Regardless of a manipulator's geometry or the amount of kinematic redundancy present in a manipulator, no fully spatial manipulator Jacobian can be equally fault tolerant to three or more joint failures.

Proof: To simplify matters, note that multiplying any of the columns of J by -1 does not affect the fault tolerance properties of J. In doing so, the corresponding columns of N^T are multiplied by -1, in which case the corresponding rows and columns of P_N are multiplied by -1. Hence, without loss of generality, we can assume that the first row and column of P_N consists of a single a followed by $n-1$ b's. Thus, for $1 < i < j \leq n$,

$$P_N \begin{pmatrix} 1 & i & j \\ 1 & i & j \end{pmatrix} = \begin{vmatrix} a & b & b \\ b & a & \pm b \\ b & \pm b & a \end{vmatrix}. \tag{22}$$

With $+b$, this becomes $a^3 - 3ab^2 + 2b^3$ and with $-b$, it becomes $a^3 - 3ab^2 - 2b^3$. These quantities are equal if and only if $b = 0$ and since $b = \frac{-1}{n}\sqrt{\frac{r(n-r)}{n-1}} \neq 0$, it follows that the various p_{ij}'s must all be equal for $1 < i < j \leq n$ for the equal fault tolerance property to hold. If $p_{ij} = b$ for $1 < i < j \leq n$ then we can write

$$P_N = (a-b)I + b \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}, \tag{23}$$

which has eigenvalues $\{a-b, \ldots, a-b, a+(n-1)b\}$. On the other hand, if $p_{ij} = -b$ for $1 < i < j \leq n$ then we can write

$$P_N = (a+b)I - b \begin{bmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & \cdots & 1 \end{bmatrix}. \tag{24}$$

In this case, the eigenvalues of P_N are $\{a+b, \ldots, a+b, a-(n-1)b\}$. Since P_N is a projection matrix, its set of eigenvalues consists of ones and zeros. There are $n-1$ eigenvalues of (24) that are equal to $a + b = \frac{r}{n} - \frac{1}{n}\sqrt{\frac{r(n-r)}{n-1}} < a < 1$. Since this quantity is bigger than zero but less than one, it follows that (24) cannot correspond to a projection matrix. Consider now the eigenvalues of (23). The $n-1$ eigenvalues that are equal to $a - b = \frac{r}{n} + \frac{1}{n}\sqrt{\frac{r(n-r)}{n-1}}$ are positive so they must equal one if (23) is a projection operator. Setting this quantity equal to one yields the result that $r = n-1$, which upon substitution into $a + (n-1)b$ yields zero. We thus conclude that P_N has rank $r = n-1$ and that the workspace dimension is $m = 1$. Hence, any J that is equally fault tolerant to three or more joint failures necessarily has the form $J = \begin{bmatrix} \pm\alpha & \cdots & \pm\alpha \end{bmatrix}$ for some $\alpha > 0$. □

As the proof indicates, Theorem 3 is applicable to any manipulator whose workspace dimension is greater than one, e.g., no planar manipulator can be equally fault tolerant to three or more failures regardless of how many joints it may have.

We now consider the case when a fully spatial manipulator is equally fault tolerant to two failures. We have already shown that this is impossible for $r = 2$. Once again, we assume without loss of generality that P_N has the form

$$P_N = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & \pm b & \cdots & \pm b \\ b & \pm b & a & \cdots & \pm b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \pm b & \pm b & \cdots & a \end{bmatrix}. \tag{25}$$

We use the property that P_N is a projection to determine restrictions on the number of degrees of redundancy that a fully spatial manipulator can have for the equal fault tolerance property to hold. As a projection, $P_N^2 = P_N$ so that for $j > 1$,

$$b = p_{1j} = (P_N)_{1j} = (P_N^2)_{1j} = 2ab + qb^2 \tag{26}$$

where q is the integer $q = n_1 - n_2 - 1$ where n_1 denotes the number of elements in the j-th column of P_N that are equal to b and n_2 denotes the number of elements equal to $-b$. Clearly $n_1 + n_2 = n - 1$ as $(P_N)_{jj} = a$ and $(P_N)_{ij} = \pm b$ for $i \neq j$. Since $b \neq 0$, (26) yields

$$q = \frac{1 - 2a}{b}. \tag{27}$$

For a redundant fully spatial manipulator, $m = 6$ and $n = r + 6$. Substituting the expressions for a and b into (27) gives

$$q = \frac{1 - \frac{2r}{n}}{\frac{-1}{n}\sqrt{\frac{r(n-r)}{n-1}}} = (r-6)\sqrt{\frac{r+5}{6r}}. \tag{28}$$

The requirement that (27) is an integer is a necessary condition for the existence of a manipulator having $r > 1$ degrees of redundancy with the property that it is equally fault tolerant to two failures.

Unfortunately, the requirement that q is an integer eliminates most if not all practical manipulator designs since only specific values of r are feasible. Indeed, it was shown in Section II-B that one can only expect to be able to design for a prescribed null space if $r \leq 12$. Testing $r = 2, 3, \ldots, 12$, one finds that only $r = 3$, 6, and 10 result in integer values of q in (28). Note that this further confirms that no fully spatial manipulator Jacobian corresponding to an 8-DOF manipulator can be equally fault tolerant to two failures. Consider now the case when $r = 3$. We have already noted that $n_1 - n_2 = q + 1$ and $n_1 + n_2 = n - 1 = r + 5$ so that $2n_1 = q + r + 6$, or, equivalently, $q + r = 2n_1 - 6$. Hence, $q + r$ is an even number so that q and r have the same parity, i.e., both are even or both are odd. However, for $r = 3$, we have $q = -2$ implying that $r = 3$ is not a feasible solution. Thus, if a redundant fully spatial manipulator with $r \leq 12$ degrees of redundancy is equally fault tolerant to two joint failures then $r = 6$ or 10.

Ten or even six degrees of redundancy would be a considerable amount of redundancy to add to a manipulator and adding that much redundancy may even make the manipulator more prone to a joint failure. So it could be argued that even if one could design a manipulator to be equally fault tolerant to two failures, it would be undesirable to do so because of the high number of degrees of redundancy required. This observation is even more significant for an orthogonal GSP. The additional requirement that JJ^T be diagonal reduces our freedom in designing a manipulator Jacobian with a prescribed null space to $9 - 2r$ degrees of freedom. For $r = 6$, this value becomes $9 - 2(6) = -3$ so that there are three more design constraints than degrees of freedom to design such a manipulator.

Note that the above argument does not conclusively prove that no fully spatial manipulator Jacobian is equally fault tolerant to two failures, but rather that if such a manipulator existed, it would require a significant, if not prohibitively high, number of degrees of redundancy.

5. CONCLUSIONS AND FUTURE WORK

In this article, the authors used relative manipulability indices to evaluate the fault tolerance of kinematically redundant manipulators to multiple joint failures. The authors provided an alternative proof of the recently proven result that the sum of the squares of the relative manipulability indices corresponding to f failures is equal to $\binom{r}{f}$. This result provides an upper bound for the worst case relative manipulability index of a manipulator with one or more failed joints. Previously, this upper bound was used to characterize optimal fault tolerance to multiple failures. However, in this article, it was shown that this upper bound is typically not achieved and is therefore not suitable for judging optimal fault tolerance. This clearly indicates the need for further consideration when designing robotics systems that are tolerant to multiple joint failures.

In the future, the authors will investigate potential methods for finding a family of 8-DOF Gough-Stewart platforms with optimal worst case fault tolerance for up to two failures by identifying the required properties of the null space of the manipulator Jacobian.

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